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Léonard GALLARDO[*] and **Chaabane REJEB**[†]

Abstract

The purpose of this paper is to present a new theory of subharmonic functions for the Dunkl-Laplace operator Δ_k in $\mathbb{R}^d$ associated to a root system and a multiplicity function $k \geq 0$. In particular, we introduce and study a Dunkl-Newton kernel and the corresponding potential of Radon measures. As applications we give a strong maximum principle, a solution of the Poisson equation and a Riesz decomposition theorem for Δ_k-subharmonic functions.

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Key words: Dunkl-Laplace operator, Generalized volume mean value operator, Dunkl subharmonic functions, Strong maximum principle, Δ_k-Riesz measure, Dunkl-Newton kernel and potentials, Riesz decomposition theorem.

Contents

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1 Introduction

Let R be a normalized root system in $\mathbb{R}^d$ i.e. R is a finite subset of $\mathbb{R}^d\backslash\{0\}$ such that for every $\alpha \in R$, $\|\alpha\| = \sqrt{2}$, $R\cap\mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha R = R$, where σ_α is the reflection with respect to the hyperplane H_α orthogonal to α (see [13] for details on root systems).

For $\xi \in \mathbb{R}^d$, let D_ξ be the Dunkl operator defined on $\mathcal{C}^1(\mathbb{R}^d)$ by

$$D_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha\in R_+} k(\alpha)\,\langle\alpha,\xi\rangle\,\frac{f(x)-f(\sigma_\alpha(x))}{\langle\alpha,x\rangle}, \tag{1.1}$$

where ∂_ξ is the ξ-directional partial derivative, R_+ is a fixed positive subsystem of R and $k : R \longmapsto [0,+\infty[$ is a fixed multiplicity function i.e. k is W-invariant, where W is the Coxeter-Weyl group generated by the reflections σ_α, $\alpha \in R$ (see [6]). These operators are related to partial derivatives by means of the Dunkl intertwining operator V_k (see [5] or [6]) as follows

$$\forall\ \xi \in \mathbb{R}^d,\quad D_\xi V_k = V_k\partial_\xi. \tag{1.2}$$

The operator V_k is a topological isomorphism from the space $\mathcal{C}^\infty(\mathbb{R}^d)$[1] onto itself satisfying (1.2) and $V_k(1) = 1$ (see [26]) and for every $x \in \mathbb{R}^d$, there exists a unique probability measure μ_x on $\mathbb{R}^d$ with compact support contained in

$$C(x) := co\{gx,\ g\in W\} \tag{1.3}$$

(the convex hull of $W.x$, the orbit of x under the group W) such that (see [22] or [23])

$$\forall\ f\in\mathcal{C}^\infty(\mathbb{R}^d),\quad V_k(f)(x) = \int_{\mathbb{R}^d} f(y)d\mu_x(y). \tag{1.4}$$

We know ([9]) that if $k > 0$, the support of μ_x is W-invariant and contains $W.x$.

The Dunkl-Laplace operator is then defined by $\Delta_k = \sum_{j=1}^d D_j^2$, where $D_j = D_{e_j}$, $j = 1,...,d$ ($(e_j)_{1\le j\le d}$ is the canonical basis of $\mathbb{R}^d$) are commuting operators (see [3] and [6]). Its action on C^2-functions is given by

$$\Delta_k f(x) = \Delta f(x) + 2\sum_{\alpha\in R_+} k(\alpha)\Big(\frac{\langle\nabla f(x),\alpha\rangle}{\langle\alpha,x\rangle} - \frac{f(x)-f(\sigma_\alpha(x))}{\langle\alpha,x\rangle^2}\Big), \tag{1.5}$$

[1] carrying its usual Fréchet topology.

where Δ (resp. ∇) is the usual Laplace (resp. gradient) operator,(see [6]).
For abbreviation, we introduce the wight function

$$\omega_k(x) := \prod_{\alpha\in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)} \tag{1.6}$$

which is W-invariant and homogeneous of degree 2γ, with the index $\gamma := \sum_{\alpha\in R_+} k(\alpha)$.

An important fact about the Dunkl-Laplace operator is that it generates a generalized heat semi-group which kernel is given by (see [21])

$$p_t(x,y) := \frac{1}{(2t)^{d/2+\gamma}c_k}\ \tau_{-x}\big(e^{-\frac{\|.\|^2}{4t}}\big)(y), \qquad x,y\in\mathbb{R}^d \tag{1.7}$$

$$:= \frac{1}{(2t)^{d/2+\gamma}c_k} e^{-(\|x\|^2+\|y\|^2)/4t} E_k\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big), \tag{1.8}$$

where $E_k(.,.)$ is the Dunkl kernel defined by $E_k(x,y) = V_k(e^{\langle \cdot, y\rangle})(x)$ (see [4], [6] and [23]), c_k is the Macdonald-Mehta constant (see [7]) given by

$$c_k := \int_{\mathbb{R}^d} \exp\big(-\tfrac{\|x\|^2}{2}\big)\omega_k(x)dx \tag{1.9}$$

and τ_x is the Dunkl translation operator which acts on $\mathcal{C}^\infty(\mathbb{R}^d)$-functions (see Annex A for precise definition and essential properties). However, note that when $f\in\mathcal{C}^\infty(\mathbb{R}^d)$ is a radial function (i.e. $f(x) = \widetilde{f}(\|x\|)$ with $\widetilde{f}$ the profile function of f), $\tau_x f$ is given by

$$\forall\ y\in\mathbb{R}^d,\quad \tau_x f(y) = \int_{\mathbb{R}^d} \widetilde{f}\big(\sqrt{\|x\|^2+\|y\|^2+2\langle x,z\rangle}\big)d\mu_y(z) \tag{1.10}$$

(see [24]). This formula shows that the Dunkl translation operators are positivity preserving on the set of radial functions.

Harmonic functions for the Dunkl-Laplacian, i.e. C^2-functions u such that $\Delta_k u = 0$, have for a long time attracted the attention of researchers involved in Dunkl theory (see ([17]), [18] and [24]) but their study was limited to C^∞-functions f defined on whole $\mathbb{R}^d$ or on the unit ball but having extension to whole $\mathbb{R}^d$.
In a recent paper ([8]), we have found a volume mean value property characterization (see below) which allows us to study Dunkl-harmonic (D-harmonic) functions on any open W-invariant subset of $\mathbb{R}^d$ (see [8] and [10]). This new approach has many benefits in particular to tackle Dunkl potential theory. It is the aim of this paper to introduce, via the heat Dunkl-semigroup and our volume mean value operator, the Dunkl-Newtonian potentials and their use to study Dunkl-subharmonic functions.
Let Ω be a W-invariant open subset of $\mathbb{R}^d$. A function $u : \Omega \longrightarrow [-\infty,+\infty[$ is called Dunkl-subharmonic (D-subharmonic) if

1. u is upper semi-continuous (u.s.c.) on Ω,

2. u is not identically $-\infty$ on each connected component of Ω,

3. u satisfies the volume sub-mean property i.e. for all closed ball $B(x,r)\subset\Omega$, we have

$$u(x) \le M_B^r(u)(x). \tag{1.11}$$

Here $M_B^r(f)(x)$ is the volume mean of f at (x,r) introduced by the authors ([8]) and defined by

$$M_B^r(f)(x) := \frac{1}{m_k(B(0,r))}\int_{\mathbb{R}^d} f(y)h_k(r,x,y)\omega_k(y)dy, \text{ [2]} \tag{1.12}$$

where m_k is the measure $dm_k(x) := \omega_k(x)dx$ and $y \mapsto h_k(r,x,y)$ is a compactly supported measurable function (see section 2) given by

$$h_k(r,x,y) := \int_{\mathbb{R}^d} \mathbf{1}_{[0,r]}(\sqrt{\|x\|^2+\|y\|^2-2\langle x,z\rangle})d\mu_y(z). \tag{1.13}$$

Of particular importance for this paper is the Dunkl type Newton kernel which is defined, when $d+2\gamma > 2$ (transient case), by means of the Dunkl heat kernel as follows

$$N_k(x,y) := \int_0^{+\infty} p_t(x,y)dt. \tag{1.14}$$

and which is finite if y is not in the W-orbit of x.

We will show that typical examples of D-subharmonic functions are the Dunkl-Newton potentials of nonpositive Radon measures. Nevertheless, in particular for lack of a non-centered Poisson kernel and because of the complexity of the Dunkl translation operators, the D-subharmonicity of these examples is not immediate and our approach to D-subharmonic functions requires some specific tools that will be presented below.

We turn now to the content and the organization of this paper. In section 2, we recall the properties of the so-called harmonic kernel $h_k(r,x,y)$ and some representation formulas involving the mean value operators.
In section 3, we study the notion of subharmonicity in Dunkl setting. In particular, we will prove that D-subharmonic functions satisfy the strong maximum principle.
The section 4 is devoted to give some characterizations of D-subharmonic functions. Here, an approximation result is the essential tool to extend the properties of C^2-D-subharmonic functions to arbitrary D-subharmonic functions.
The notion of Riesz measure of a D-subharmonic function will be introduced in section 5. We will study the Dunkl type Newton kernel and potential of a Radon measure on $\mathbb{R}^d$ in section 6. In particular, we will discuss the D-harmonicity and the D-superhamonicity of these objects and we will obtain the mass uniqueness principle.
Finally, in section 7, we prove a Riesz decomposition theorem for D-subharmonic functions and we describe all bounded from above D-subharmonic functions in the whole space.

Notations: Let us introduce the following functional spaces and notations which will be used throughout the paper. For Ω a W-invariant open subset of $\mathbb{R}^d$, we denote by:
• $L^1_{k,loc}(\Omega) = L^1_{loc}(\Omega,\ m_k)$ the space of measurable functions $f : \Omega \longrightarrow \mathbb{C}$ such that $\int_K |f(x)|\omega_k(x)dx < +\infty$ for any compact set $K \subset \Omega$.
• $\mathcal{D}(\Omega)$ the space of C^∞-functions on Ω with compact support.

[2] Note that if the function f is u.s.c, then f is bounded from above on compact sets and $M_B^r(f)(x)$ is well-defined (eventually equal to $-\infty$).

• $\mathcal{D}'(\Omega)$ the space of distributions on Ω (i.e. the topological dual of $\mathcal{D}(\Omega)$ carrying the Fréchet topology).
• $\mathcal{M}^+(\mathbb{R}^d)$ the set of nonnegative Radon measures on $\mathbb{R}^d$.
• $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of C^∞-functions on $\mathbb{R}^d$ which are rapidly decreasing together with their derivatives.
• $B(a,\rho)$ $\big($resp. $\overset{\circ}{B}(a,\rho)$, resp. $B^W(a,\rho) := \cup_{g\in W} B(ga,\rho)\big)$ the closed Euclidean $\big($resp. the open Euclidean, resp. the closed Dunkl$\big)$ ball centered at a and with radius $\rho > 0$.

2 The harmonic kernel and the mean value operators

2.1 Properties of the harmonic kernel

Let $(r,x,y) \mapsto h_k(r,x,y)$ the harmonic kernel defined by (1.13). We note that in the classical case (i.e. $k=0$), we have $\mu_y = \delta_y$ and $h_0(r,x,y) = \mathbf{1}_{[0,r]}(\|x-y\|) = \mathbf{1}_{B(x,r)}(y)$. The harmonic kernel satisfies the following properties (see [8]):

1. For all $r>0$ and $x,y \in \mathbb{R}^d$, $0 \le h_k(r,x,y) \le 1$.

2. For all fixed $x,y \in \mathbb{R}^d$, the function $r \longmapsto h_k(r,x,y)$ is right-continuous and non decreasing on $]0,+\infty[$.

3. For all fixed $r>0$ and $x \in \mathbb{R}^d$,
$$B(x,r) \subset \text{supp } h_k(r,x,.\,) \subset B^W(x,r) := \cup_{g\in W} B(gx,r). \tag{2.1}$$
The first inclusion is proved in [9] while the second one is proved in [8].

4. Let $r>0$ and $x \in \mathbb{R}^d$. For any sequence $(\chi_\varepsilon) \subset \mathcal{D}(\mathbb{R}^d)$ of radial functions such that for every $\varepsilon > 0$, $0 \le \chi_\varepsilon \le 1$, $\chi_\varepsilon = 1$ on $B(0,r)$ and $y \in \mathbb{R}^d$, $\lim_{\varepsilon\to 0} \chi_\varepsilon(y) = \mathbf{1}_{B(0,r)}(y)$, we have
$$\forall\ y \in \mathbb{R}^d, \quad h_k(r,x,y) = \lim_{\varepsilon\to 0} \tau_{-x}\chi_\varepsilon(y). \tag{2.2}$$

5. For all $r>0$, all $x,y \in \mathbb{R}^d$ and all $g \in W$, we have
$$h_k(r,x,y) = h_k(r,y,x) \quad \text{and} \quad h_k(r,gx,y) = h_k(r,x,g^{-1}y). \tag{2.3}$$

6. For all $r>0$ and $x \in \mathbb{R}^d$, we have
$$\|h_k(r,x,.)\|_{k,1} := \int_{\mathbb{R}^d} h_k(r,x,y)\omega_k(y)dy = m_k(B(0,r)) = \frac{d_k r^{d+2\gamma}}{d+2\gamma}, \tag{2.4}$$
where d_k is the constant
$$d_k := \textstyle\int_{S^{d-1}} \omega_k(\xi)d\sigma(\xi) = \frac{c_k}{2^{d/2+\gamma-1}\Gamma(d/2+\gamma)}. \tag{2.5}$$
Here $d\sigma(\xi)$ is the surface measure of the unit sphere S^{d-1} of $\mathbb{R}^d$ and c_k is defined in (1.9).

7. Let $r > 0$ and $x \in \mathbb{R}^d$. Then the function $h_k(r, x, .)$ is upper semi-continuous on $\mathbb{R}^d$.

8. The harmonic kernel satisfies the following fundamental geometric inequality: if $\|a - b\| \leq 2r$ with $r > 0$, then

$$\forall\, \xi \in \mathbb{R}^d, \quad h_k(r, a, \xi) \leq h_k(4r, b, \xi). \tag{2.6}$$

Note that if $k = 0$, this inequality says that if $\|a - b\| \leq 2r$, then $B(a, r) \subset B(b, 4r)$.

9. Let $x \in \mathbb{R}^d$. Then the family of probability measures

$$d\eta^k_{x,r}(y) = \frac{1}{m_k[B(0, r)]} h_k(r, x, y)\omega_k(y)dy \tag{2.7}$$

is an approximation of the Dirac measure δ_x as $r \longrightarrow 0$. That is

$$\forall\, \alpha > 0, \quad \lim_{r \to 0} \int_{\|x-y\|>\alpha} d\eta^k_{x,r}(y) = 0 \tag{2.8}$$

and if f is a continuous function on a W-invariant open neighborhood of x, then (see [8], Proposition 3.2):

$$\lim_{r \to 0} \int_{\mathbb{R}^d} f(y) d\eta^k_{x,r} = \lim_{r \to 0} M^r_B(f)(x) = f(x). \tag{2.9}$$

Let Ω be a W-invariant nonempty open subset of $\mathbb{R}^d$. The boundeness of h_k as well as its support property (2.1) allowed us to define the volume mean of any $f \in L^1_{k,loc}(\Omega)$ at (x, r) by (1.12) whenever $B(x, r) \subset \Omega$. We will need the following notations which will be used frequently in this paper:

$$\forall\, r > 0, \quad \Omega_r := \big\{x \in \Omega; \;\; dist(x, \partial\Omega) > r\big\}, \tag{2.10}$$

$$r_\Omega := \sup\{r > 0; \quad \Omega_r \neq \emptyset\}. \tag{2.11}$$

Clearly, we have $\Omega_{r_1} \subset \Omega_{r_2}$ whenever $r_2 \leq r_1$ and $\Omega = \cup_{r>0}\Omega_r = \cup_{r<r_\Omega}\Omega_r$. Moreover, since $\Omega_r = \big\{x \in \Omega; \;\; B(x, r) \subset \Omega\big\}$, the open set Ω_r, $r < r_\Omega$, is W-invariant.
The volume mean operator of $f \in L^1_{k,loc}(\Omega)$ has the following properties (the first and the second results are proved in [10] while the third is proved in [19]):

Proposition 2.1 *Let* $f \in L^1_{k,loc}(\Omega)$.

1. *Let* $r < r_\Omega$. *Then the function* $M^r_B(f)$ *belongs to* $L^1_{k,loc}(\Omega_r)$.

2. *Let* $x \in \Omega$. *Then the function* $r \mapsto M^r_B(f)(x)$ *is continuous on* $]0, \varrho_x[$ *with*

$$\varrho_x := dist(x, \partial\Omega). \tag{2.12}$$

3. *For almost every*[3] $x \in \Omega$, *we have* $\lim_{r\to 0} M^r_B(f)(x) = f(x)$.

[3]Note that negligible sets for the Lebesgue measure coincide with negligible sets for the measure m_k.

2.2 Representation formulas for the mean value operators

In this subsection, we will recall some representation formulas obtained by the authors in [8] and [10]. These formulas play a key role in the study of D-subharmonic functions in sections 4, 5 and 7.
Let us begin to recall that the spherical mean for C^∞-functions defined on whole $\mathbb{R}^d$ as follows (see [18])

$$M_S^r(f)(x) := \frac{1}{d_k}\int_{S^{d-1}} \tau_x f(ry)\omega_k(y)d\sigma(y)^4. \tag{2.13}$$

It is shown in [24] that there exists a compactly supported probability measure $\sigma_{x,r}^k$ on $\mathbb{R}^d$ such that the spherical mean of $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ at $(x,r) \in \mathbb{R}^d \times \mathbb{R}_+$ is given by

$$M_S^r(f)(x) = \int_{\mathbb{R}^d} f(y)d\sigma_{x,r}^k(y), \tag{2.14}$$

with

$$\text{supp } \sigma_{x,r}^k \subset B^W(x,r) = \cup_{g\in W} B(gx,r). \tag{2.15}$$

Formula (2.14) shows that we can define the spherical mean at (x,r) of any measurable nonnegative (resp. nonpositive, resp. bounded) function on $B^W(x,r)$.

The following crucial results, proved by the authors, on the link between the spherical and volume means hold: If $f \in \mathcal{C}^2(\Omega)$, then for every closed ball $B(x,r) \subset \Omega$, $r > 0$, we have:

$$M_S^r(f)(x) = f(x) + \frac{1}{d+2\gamma}\int_0^r M_B^t(\Delta_k f)(x)\ t dt, \tag{2.16}$$

and

$$M_B^r(f)(x) = f(x) + \frac{1}{r^{d+2\gamma}}\int_0^r \int_0^\rho M_B^t(\Delta_k f)(x)\ t\ dt\ \rho^{d+2\gamma-1} d\rho. \tag{2.17}$$

Note that (2.16) and (2.17) have been proved at first for $C^\infty(\mathbb{R}^d)$-functions in [8] and then have been extended by the authors to $C^2(\Omega)$-functions using approximation results (see [10] for more details).
Furthermore, the following relation holds for continuous functions on Ω (see [10])

$$M_B^r(f)(x) = \frac{d+2\gamma}{r^{d+2\gamma}}\int_0^r M_S^t(f)(x)t^{d+2\gamma-1}dt, \quad \text{whenever} \quad B(x,r) \subset \Omega. \tag{2.18}$$

Now, let f be an upper semi-continuous (u.s.c.) function on Ω and let $B(x,r) \subset \Omega$. As f is u.s.c., by adding a constant, we can assume that f is nonpositive on the compact set $B^W(x,r)$. Therefore, using (2.1) and (2.15), we can define the Dunkl-volume and the Dunkl-spherical means of f relative to (x,r). Moreover, we have

Lemma 2.1 *The relation (2.18) holds for the u.s.c. function f on Ω (the two terms of (2.18) being eventually equal to $-\infty$).*

[4] Recalling that $d\sigma$ is the surface measure on the unit sphere S^{d-1} of $\mathbb{R}^d$.

Proof: Fix $x \in \Omega$ and $r > 0$ such that $B(x,r) \subset \Omega$. Since f is bounded from above on $B^W(x,r)$, there is a decreasing sequence of continuous functions (f_n) such that $f_n \longrightarrow f$ pointwise on $B^W(x,r)$. Replacing f_n by $f_n - \sup_{B^W(x,r)} f_1$ and f by $f - \sup_{B^W(x,r)} f_1$, we may assume that f and all f_n are nonpositive on $B^W(x,r)$.
For $t \in]0,r]$, set $g_n(t) = M_S^t(f_n)(x)$ and $g(t) = M_S^t(f)(x)$. We can see that the sequence (g_n) is decreasing and from the monotone convergence theorem applied to the sequence (f_n), we get $g_n \longrightarrow g$ pointwise on $]0,r]$ and in particular, g is a measurable function.
Let us now apply the monotone convergence theorem to the sequence (g_n), we obtain

$$\int_0^r M_S^t(f)(x)t^{2\gamma+d-1}dt = \lim_{n\longrightarrow+\infty} \int_0^r M_S^t(f_n)(x)t^{2\gamma+d-1}dt. \tag{2.19}$$

But, by the first step,

$$\frac{2\gamma+d}{r^{2\gamma+d}} \int_0^r M_S^t(f_n)(x)t^{2\gamma+d-1}dt = M_B^r(f_n)(x) \tag{2.20}$$

and once again by the monotone convergence theorem, we have

$$\lim_{n\longrightarrow+\infty} M_B^r(f_n)(x) = M_B^r(f)(x). \tag{2.21}$$

Finally, we deduce the relation (2.18) from (2.19), (2.20) and (2.21). □

3 Dunkl subharmonic functions

In this section, we study some properties of D-subharmonic functions (see definition (1.11)) on a W-invariant open set $\Omega \subset \mathbb{R}^d$. In particular, we will prove that they satisfy the strong maximum principle and the uniqueness principle.

Let us denote by $\mathcal{SH}_k(\Omega)$ the set of D-subharmonic functions on Ω which is clearly a convex cone. Furthermore, it is not difficult to see that if $u, v \in \mathcal{SH}_k(\Omega)$ and if f is a convex and non-decreasing function on $\mathbb{R}$, then $\max(u,v)$ and $f(u)$ are also in $\mathcal{SH}_k(\Omega)$. As in the classical case, a function u is called D-superharmonic if $-u$ is D-subharmonic.

3.1 Local properties of D-subharmonic functions

Proposition 3.1 *Let $u \in \mathcal{SH}_k(\Omega)$. Then the function u belongs to $L^1_{k,loc}(\Omega)$.*

Proof: Fix Ω_0 a connected component of Ω. Let

$$E := \{x \in \Omega_0, \ \ u\omega_k \ \text{ is integrable over some neighbourhood of } x\}.$$

Let $x \in E$. Then there exists $r > 0$ such that $B(x,r) \subset \Omega_0$ and $\int_{B(x,r)} |u(y)|\omega_k(y)dy < +\infty$. For $z \in B(x,r/2)$, we have $B(z,r/2) \subset B(x,r)$ and hence $u\omega_k$ is integrable over $B(z,r/2)$. Thus, $B(x,r/2) \subset E$ and E is an open subset of Ω_0.
Now, let $x \in \Omega_0 \backslash E$. Because $u\omega_k$ is not integrable on any neighborhood of x, we must have $\int_{B(x,R)} |u(y)|\omega_k(y)dy = +\infty$ for all $R > 0$ such that $B(x,R) \subset \Omega_0$. Fix $r > 0$ such that $B(x,6r) \subset \Omega_0$. We will prove that $B(x,2r) \subset \Omega_0 \backslash E$.
Since u is u.s.c., we can assume that u is nonpositive on the compact set $K = B^W(x,6r)$[5].

[5]replacing u by $u - \max_K u$.

Let $z \in B(x, 2r)$. From (2.6) and the nonpositivity of u, we deduce that

$$\int_{\mathbb{R}^d} u(y)h_k(4r, z, y)\omega_k(y)dy \leq \int_{\mathbb{R}^d} u(y)h_k(r, x, y)\omega_k(y)dy. \tag{3.1}$$

Now, if we apply (2.6) once again where we replace respectively r, a, b and ξ by $r/4$, x, y and x we get

$$\forall\ y \in B(x, r/2), \quad h(r/4, x, x) \leq h_k(r, y, x) \tag{3.2}$$

Thus, using (3.2), (2.3), (3.1), (2.1) and the fact that $u \leq 0$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} u(y)h_k(4r, z, y)\omega_k(y)dy &\leq \int_{B(x,r/2)} u(y)h_k(r, x, y)\omega_k(y)dy \\ &\leq h_k(r/4, x, x) \int_{B(x,r/2)} u(y)\omega_k(y)dy = -\infty.\end{aligned}$$

Consequently, from the previous inequality we get $M_B^{4r}(u)(z) = -\infty$, and therefore, $u(z) = -\infty$ by the sub-mean property. Hence, $u = -\infty$ on $B(x, 2r)$ and this proves that $\Omega_0 \backslash E$ is an open subset of Ω_0. Finally, as $u \neq -\infty$ on Ω_0 and using the connectedness of Ω_0, we must have $E = \Omega_0$. The connected component Ω_0 being arbitrary, Proposition 3.1 is proved. $\square$

Let $u \in \mathcal{SH}_k(\Omega)$. Using the generalized Lebesgue differentiation theorem (see [19]) and Proposition 3.1, we have $u(x) = \lim_{r\to 0} M_B^r(u)(x)$ for almost all $x \in \Omega$.
In the classical case (i.e. when $k = 0$), this equality holds everywhere for any subharmonic function (see for example [1], Corollary 3.2.6 or [12], Lemma 2.4.4). In the following result, we will extend this fundamental property to D-subharmonic functions.

Proposition 3.2 *Let $u \in \mathcal{SH}_k(\Omega)$. Then, for every $x \in \Omega$, we have*

$$u(x) = \lim_{r\to 0} M_B^r(u)(x). \tag{3.3}$$

Proof: Fix $x \in \Omega$ and $R > 0$ such that $B(x, R) \subset \Omega$. As above, we may assume that u is negative on the compact set $B^W(x, R)$. We distinguish two cases:
`First case:` Suppose that $u(x) > -\infty$. By upper semi-continuity, for all $\varepsilon > 0$, there exists $\alpha \in]0, R]$ such that

$$u(y) < u(x) + \varepsilon, \quad \text{whenever } y \in B(x, \alpha). \tag{3.4}$$

From the sub-mean property and the fact that $u < 0$ on $B^W(x, R)$, we have

$$\forall\ r \in]0, R], \quad u(x) \leq M_B^r(u)(x) = \int_{\mathbb{R}^d} u(y)d\eta_{x,r}^k(y) \leq \int_{B(x,\alpha)} u(y)d\eta_{x,r}^k(y),$$

where $d\eta_{x,r}^k(y)$ is the probability measure defined by (2.7).
Using (3.4), we deduce that

$$\forall\ r \in]0, R], \quad u(x) \leq M_B^r(u)(x) \leq (u(x) + \varepsilon) \int_{B(x,\alpha)} d\eta_{x,r}^k(y). \tag{3.5}$$

As from (2.8) $\lim_{r\to 0}\int_{B(x,\alpha)} d\eta^k_{x,r}(y) = 1$, there exists $\beta \in]0, R[$ such that

$$\forall\ r \in]0,\beta], \quad \int_{B(x,\alpha)} d\eta^k_{x,r}(y) \geq 1-\varepsilon. \tag{3.6}$$

Now, if we have taken $\varepsilon > 0$ small enough to ensure that $u(x)+\varepsilon < 0$, we deduce from (3.5) and (3.6) that

$$\forall\ r \in]0,\beta], \quad u(x) \leq M^r_B(u)(x) \leq u(x) + \varepsilon(1-\varepsilon-u(x)).$$

This implies that $M^r_B(u)(x) \longrightarrow u(x)$ as $r \longrightarrow 0$. This proves the result in this case.
`Second case:` Suppose that $u(x) = -\infty$. For every $n \in \mathbb{N}\backslash\{0\}$, there is $a \in]0, R]$ such that $u(y) \leq -n$ whenever $y \in B(x,a)$. Therefore,

$$\forall\ r \in]0,a], \quad M^r_B(u)(x) \leq -n\int_{B(x,a)} d\eta^k_{x,r}(y). \tag{3.7}$$

Again by (2.8), there exists $b > 0$ such that

$$\forall\ r \in]0,b], \quad \int_{B(x,a)} d\eta^k_{x,r}(y) \geq 1/2. \tag{3.8}$$

From (3.7) and (3.8) we obtain $\forall\ r \in]0,\min(a,b)],\quad M^r_B(u)(x) \leq -n/2$. Therefore, $M^r_B(u)(x) \longrightarrow -\infty$ as $r \longrightarrow 0$ and the result is also proved in this case.

□

From the previous Proposition, we immediately obtain the uniqueness principle that a D-subharmonic function is determined by its restriction to the complementary of a negligible set. More precisely:

Corollary 3.1 *If u and v are D-subharmonic functions on a W-invariant open set $\Omega \subset \mathbb{R}^d$ and $u(x) = v(x)$ for almost every $x \in \Omega$, then u and v are identically equal in Ω.*

In the following result we consider the convergence property of a decreasing sequence of D-subharmonic functions.

Proposition 3.3 *Let (u_n) be a decreasing sequence of D-subharmonic functions on Ω and $u(x) := \lim_{n\to+\infty} u_n(x)$. If u is not identically $-\infty$ on each connected component of Ω, then u is D-subharmonic on Ω.*

Proof: Clearly u is u.s.c. on Ω as being a decreasing limit of u.s.c. functions. Let $x \in \Omega$ and $r > 0$ such that $B(x,r) \subset \Omega$. By the monotone convergence theorem, we get

$$u(x) = \lim_{n\to+\infty} u_n(x) \leq \lim_{n\to+\infty} M^r_B(u_n)(x) = M^r_B(u)(x).$$

This implies that u is D-subharmonic on Ω. □

3.2 The strong Maximum principle

The following theorem is a generalization of the strong maximum principle for D-harmonic functions obtained by the authors in [8] (Theorem 4.1).

Theorem 3.1 *Let $u \in \mathcal{SH}_k(\Omega)$ and suppose that Ω is connected.*

i) *If u has a maximum in Ω, then u is constant.*

ii) *If Ω is bounded and $\limsup_{z\to x} u(z) \le 0$, for all $x \in \partial\Omega$, then $u \le 0$ on Ω.*

Proof: **i)** Let $x_0 \in \Omega$ such that $u(x) \le u(x_0)$ for all $x \in \Omega$. Let

$$\Omega_0 := \{x \in \Omega, \quad u(x) < u(x_0)\}.$$

Because u is u.s.c., Ω_0 is an open subset of Ω.
Now, let $x \in \Omega\backslash\Omega_0$ i.e. $u(x) = u(x_0)$ and $r > 0$ such that $B(x,r) \subset \Omega$. By the sub-mean property, we clearly have

$$u(x_0) = u(x) \le M_B^r(u)(x) \le u(x_0).$$

This yields

$$\frac{1}{m_k(B(0,r))}\int_{\mathbb{R}^d}[u(x_0) - u(y)]h_k(r,x,y)\omega_k(y)dy = 0.$$

Hence, $u(x_0) = u(y)$ for almost every $y \in \text{supp } h_k(r,x,.)$ and by (2.1), $u(x_0) = u(y)$ for almost every $y \in \overset{\circ}{B}(x,r)$. Let us now introduce the nonpositive function $v(y) = u(y) - u(x_0)$, $y \in \overset{\circ}{B}(x,r)$. Suppose that there exists $a \in \overset{\circ}{B}(x,r)$ such that $v(a) < 0$ and take $\lambda \in \mathbb{R}$ such that $v(a) < \lambda < 0$. Since v is u.s.c at the point a, there is $\epsilon > 0$ such that $B(a,\epsilon) \subset \overset{\circ}{B}(x,r)$ and $v(y) < \lambda$ for all $y \in B(a,\epsilon)$. This contradicts the fact that $v = 0$ a.e. on $\overset{\circ}{B}(x,r)$ and this proves that $u \equiv u(x_0)$ on $\overset{\circ}{B}(x,r)$.
Consequently, $\Omega\backslash\Omega_0$ is an open subset of Ω containing x_0. But Ω is connected, then $\Omega_0 = \emptyset$ and this shows i).
ii) Define the function $\widetilde{u}$ on the compact closure $\overline{\Omega}$ of Ω by $\widetilde{u}(x) = u(x)$ if $x \in \Omega$ and $\widetilde{u}(x) = \limsup_{y\to x,\ y\in\Omega} u(y)$ if $x \in \partial\Omega$.
Clearly $\widetilde{u}$ is u.s.c. on $\overline{\Omega}$. Consequently, there exists $x_0 \in \overline{\Omega}$ such that $\widetilde{u}(x_0) = \sup_{\overline{\Omega}} \widetilde{u}(x)$.
If $\widetilde{u}(x_0) > 0$, then by our hypothesis necessarily $x_0 \in \Omega$ and by *i)* we have $u(x) = u(x_0) > 0$ for every $x \in \Omega$. We obtain a contradiction to the fact that $\limsup_{y\to x} u(y) \le 0$. □

Corollary 3.2 *Let $u \in \mathcal{SH}_k(\Omega)$ and suppose that G is a connected W-invariant open subset of Ω with compact closure $\overline{G} \subset \Omega$. If s is D-superharmonic on Ω and $u \le s$ on ∂G, then $u \le s$ on G.*

Proof: Clearly $u - s$ is D-subharmonic on G and for $x \in \partial G$, we have

$$\limsup_{z\to x}[u(z) - s(z)] \le \limsup_{z\to x} u(z) - \liminf_{z\to x} s(z) = u(x) - s(x) \le 0.$$

Hence, the result follows from Theorem 3.1, ii). □

4 Characterization of Dunkl subharmonic functions

Our aim in this section is give some characterizations of the Δ_k-subharmonicity. We will first do this for C^2-Δ_k-subharmonic functions. Then, an approximation method allowed us to extend the results to any Δ_k-subharmonic function.

4.1 Characterization of C^2- D-subharmonic functions

As a first result, we have

Proposition 4.1 *Let $u \in \mathcal{C}^2(\Omega)$. Then the following assertions are equivalent*

i) *$u \in \mathcal{SH}_k(\Omega)$,* ***ii)*** *$\Delta_k u \geq 0$ on Ω,* ***iii)*** *$u(x) \leq M_S^r(u)(x)$ whenever $B(x,r) \subset \Omega$.*

Proof: **i) $\Longrightarrow$ ii)** Suppose that $\Delta_k u(x) < 0$ for some $x \in \Omega$. By (2.9), we have $\lim_{t\to 0} M_B^t(\Delta_k u)(x) = \Delta_k u(x)$. Hence, there exists $r \in]0, \varrho_x[$ such that[6]

$$M_B^t(\Delta_k u)(x) \leq \frac{1}{2}\Delta_k u(x) < 0 \quad \text{for all} \quad t \in]0, r].$$

This implies that

$$\frac{1}{r^{2\gamma+d}} \int_0^r \int_0^\rho M_B^t(\Delta_k u)(x)\, t\, dt\, \rho^{2\gamma+d-1} d\rho \leq \frac{r^2}{4(d+2\gamma+2)}\Delta_k u(x) < 0.$$

Therefore, by (2.17) we obtain $M_B^r(u)(x) < u(x)$. A contradiction with the sub-mean property.
ii) $\Longrightarrow$ iii) This follows immediately from the relation (2.16).
iii) $\Longrightarrow$ i) From (2.18) and a direct integration with respect to r, we obtain the result. □

The C^2- D-subharmonicity can be characterized in terms of the monotonicity with respect to r of the spherical and volume means. More precisely, we have

Proposition 4.2 *Let $u \in \mathcal{C}^2(\Omega)$. The following statements are equivalent*

i) *$u \in \mathcal{SH}_k(\Omega)$,*

ii) *for every $x \in \Omega$, the function $r \longmapsto M_B^r(u)(x)$ is non-decreasing on $]0, \varrho_x[$ and*

$$\lim_{r\to 0} M_B^r(u)(x) = u(x), \tag{4.1}$$

iii) *for every $x \in \Omega$, the function $r \longmapsto M_S^r(u)(x)$ is non-decreasing on $]0, \varrho_x[$ and*

$$\lim_{r\to 0} M_S^r(u)(x) = u(x), \tag{4.2}$$

iv) *$u \in L^1_{k,loc}(\Omega)$, $\lim_{r\to 0} M_B^r(u)(x) = u(x)$ for every $x \in \Omega$ and $M_B^r(u)(x) \leq M_S^r(u)(x)$, whenever $B(x,r) \subset \Omega$.*

[6] We recall that ϱ_x is the distance from x to the boundary of Ω (see (2.12)).

Proof: At first, using Proposition 2.1- 2), formulas (2.16) and (2.17), we deduce that the functions $r \longmapsto M_B^r(f)(x)$ and $r \longmapsto M_S^r(f)(x)$ are differentiable on $]0, \varrho_x[$ and the relations (4.1) and (4.2) are always satisfied for any fixed function $f \in \mathcal{C}^2(\Omega)$ and for any fixed $x \in \Omega$. We note also that the first condition in assertion iv) is redundant but we will need it in order to extend this result to an arbitrary D-subharmonic function (see Theorem 4.2 below).
ii) $\Longrightarrow$ **i)** As $r \longmapsto M_B^r(u)(x)$ is non-decreasing, (4.1) implies that the sub-mean property is clearly satisfied.
i) $\Longrightarrow$ **iii)** We use the fact that $\Delta_k u \geq 0$ on Ω and we differentiate with respect to r the relation (2.16) and we get $\frac{d}{dr} M_S^r(u)(x) \geq 0$ i.e we obtain iii).
iii) $\Longrightarrow$ **iv)** It is a direct consequence of the relation (2.18).
iv) $\Longrightarrow$ **ii)** We differentiate with respect to r in the relation (2.18) and we obtain

$$\frac{d}{dr} M_B^r(u)(x) = \frac{d+2\gamma}{r}\big(M_S^r(u)(x) - M_B^r(u)(x)\big) \geq 0.$$

This implies that $r \longmapsto M_B^r(u)(x)$ is non-decreasing on $]0, \varrho_x[$. $\square$

4.2 Approximation of D-subharmonic functions by $\mathcal{C}^\infty$-functions

Let us consider the following radial function $\varphi(x) := a \exp\left(-\frac{1}{1-\|x\|^2}\right)\mathbf{1}_{B(0,1)}(x)$, $x \in \mathbb{R}^d$, where a is a constant such that $x \longmapsto \varphi(x)\omega_k(x)$ is a probability density.
For $\varepsilon > 0$, define the function

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^{d+2\gamma}}\varphi\left(\frac{x}{\varepsilon}\right). \tag{4.3}$$

It is well known that $\varphi_\varepsilon \in \mathcal{D}(\mathbb{R}^d)$ is radial with supp $\varphi_\varepsilon \subset B(0, \varepsilon)$.

In order to approximate D-subharmonic functions by smooth D-subharmonic functions, we need the following facts, proved in [10], on Dunkl convolution product

Proposition 4.3 *Let $u \in L^1_{k,loc}(\Omega)$ and r_Ω given by (2.11). For $0 < \varepsilon < r_\Omega$, define the function u_ε by*

$$\forall\ x \in \Omega_\varepsilon, \quad u_\varepsilon(x) := u *_k \varphi_\varepsilon(x) := \int_{\mathbb{R}^d} u(y)\tau_{-x}\varphi_\varepsilon(y)\omega_k(y)dy. \tag{4.4}$$

Then the sequence $(u_\varepsilon)_{\varepsilon < r_\Omega}$ satisfies

i) *For every $\varepsilon < r_\Omega$, the function u_ε is in $\mathcal{C}^\infty(\Omega_\varepsilon)$ and we have*

$$\Delta_k u_\varepsilon(x) = \Delta_k(u *_k \varphi_\varepsilon)(x) = u *_k \Delta_k\varphi_\varepsilon(x), \quad x \in \Omega_\varepsilon. \tag{4.5}$$

ii) *For every $\varepsilon < r_\Omega$ and every closed ball $B(x, r) \subset \Omega_\varepsilon$, we have*

$$M_B^r(u_\varepsilon)(x) := M_B^r(u *_k \varphi_\varepsilon)(x) = M_B^r(u) *_k \varphi_\varepsilon(x). \tag{4.6}$$

iii) *For almost every $x \in \Omega$, $u_\varepsilon(x) \longrightarrow u(x)$ as $\varepsilon \longrightarrow 0$.*

iv) *If u is continuous on Ω, then for every $x \in \Omega$, $u_\varepsilon(x) \longrightarrow u(x)$ as $\varepsilon \longrightarrow 0$.*

Moreover, the following associativity result

$$\big(u *_k \varphi_{\varepsilon_1}\big) *_k \varphi_{\varepsilon_2} = \big(u *_k \varphi_{\varepsilon_2}\big) *_k \varphi_{\varepsilon_1}, \quad on \quad \Omega_{\varepsilon_1+\varepsilon_2}, \tag{4.7}$$

holds whenever $\varepsilon_1 + \varepsilon_2 < r_\Omega$ (see [10], Proposition 3.3).

Remark 4.1 *In order to prove that $u *_k \varphi_\varepsilon$ is well defined on Ω_ε, we have used the following support property (see [10])*

$$supp\ \tau_{-x}\varphi_\varepsilon \subset B^W(x,\varepsilon) = \cup_{g\in W} B(gx,\varepsilon). \tag{4.8}$$

Our approximate result is as follows:

Theorem 4.1 *Let $u \in \mathcal{SH}_k(\Omega)$ and u_ε the functions defined by (4.4). Then we have*

1) *for every $0 < \varepsilon < r_\Omega$, the function u_ε is D-subharmonic and of class C^∞ on Ω_ε,*

2) *for every $0 < \rho < r_\Omega$, the sequence $(u_\varepsilon)_{0<\varepsilon<\rho}$ of C^∞ and D-subharmonic functions on Ω_ρ is non-decreasing [7] and converges pointwise to u on Ω_ρ as $\varepsilon \to 0$,*

3) *for all $B(x,r) \subset \Omega$, $M_B^r(u_\varepsilon)(x) \longrightarrow M_B^r(u)(x)$ and $M_S^r(u_\varepsilon)(x) \longrightarrow M_S^r(u)(x)$ as $\varepsilon \to 0$.*

Proof: **1)** By Proposition 3.1, $u \in L^1_{k,loc}(\Omega)$ and then from Proposition 4.3 we deduce that $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon)$. On the other hand, as u is D-subharmonic on Ω and $\tau_{-x}\varphi_\varepsilon \geq 0$, (4.6) implies that

$$M_B^r(u_\varepsilon)(x) \geq u_\varepsilon(x), \quad \text{for all } B(x,r) \subset \Omega_\varepsilon.$$

Therefore, u_ε is D-subharmonic on Ω_ε.
2) Choose $0 < \rho < r_\Omega$ (i.e. Ω_ρ is nonempty). By 1) and i) of Proposition 4.3, we have $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\rho) \cap \mathcal{SH}_k(\Omega_\rho)$ for all $\varepsilon < \rho$.
• We will prove in two steps that the sequence $(u_\varepsilon)_{0<\varepsilon<\rho}$ is non-decreasing.
`Step1:` Suppose that u is of class C^2 on Ω. According to [10] (see Proposition 3.2), the relation (4.4) can be rewritten in spherical coordinates as follows

$$u *_k \varphi_\varepsilon(x) = d_k \int_0^\varepsilon \widetilde{\varphi_\varepsilon}(t) t^{d+2\gamma-1} M_S^t(u)(x)dt, \tag{4.9}$$

where $\widetilde{\varphi_\varepsilon}$ is the profile function of φ_ε and d_k is the constant given by (2.5). Using the change of variables $\theta = t/\varepsilon$ in (4.9) and recalling (4.3), we deduce that

$$u_\varepsilon(x) = d_k \int_0^1 \widetilde{\varphi}(\theta)\ \theta^{d+2\gamma-1} M_S^{\theta\varepsilon}(u)(x)d\theta.$$

Since, $r \longmapsto M_S^r(u)(x)$ is non-decreasing (see Proposition 4.2), we conclude that $(u_\varepsilon)_{0<\varepsilon<\rho}$ is a non-decreasing sequence.
`Step 2:` Suppose only that $u \in \mathcal{SH}_k(\Omega)$. In order to use the same idea many times in the sequel of this paper, we will present the argument in the form of the following fundamental approximation lemma:

[7] i.e. for all fixed $x \in \Omega_\rho$, $\varepsilon \mapsto u_\varepsilon(x)$ is a non-decreasing function on $]0,\rho[$.

Lemma 4.1 *Let $v \in L^1_{k,loc}(\Omega)$ and (φ_ε) the sequence defined by (4.3). Assume that for any $\varepsilon < r_\Omega$, the function $v *_k \varphi_\varepsilon$ belongs to $\mathcal{SH}_k(\Omega_\varepsilon)$. Then*

a) *for every $0 < \rho < r_\Omega$, the sequence $(v *_k \varphi_\varepsilon)_{0<\varepsilon<\rho}$ is non-decreasing on Ω_ρ,*

b) *the function $s : x \mapsto \lim_{\varepsilon\to 0} v *_k \varphi_\varepsilon(x)$ is well defined and D-subharmonic on Ω and $v = s$ almost everywhere on Ω.*

Assume the result of the Lemma for the moment.
By Proposition 3.1 and the statement 1) of the theorem, the hypotheses of Lemma 4.1 are satisfied. Consequently, using Lemma 4.1 and the uniqueness principle (Corollary 3.1), we obtain the statement 2).
3) By 2), the result follows immediately from the monotone convergence theorem. □

Proof of Lemma 4.1: **a)** Fix $\rho \in]0, r_\Omega[$ and let $\eta \in]0, \rho[$. By our hypothesis and Proposition 4.3, the function $v *_k \varphi_\eta \in \mathcal{C}^\infty(\Omega_\rho) \cap \mathcal{SH}_k(\Omega_\rho)$. Consequently, by the statement 1) of Theorem 4.1, the functions $[v *_k \varphi_\eta] *_k \varphi_\varepsilon$, with $\varepsilon > 0$ such that $\eta + \varepsilon < \rho$, are in $\mathcal{C}^\infty(\Omega_\rho) \cap \mathcal{SH}_k(\Omega_\rho)$. Furthermore, by the step 1, the sequence $\big([v *_k \varphi_\eta] *_k \varphi_\varepsilon\big)_{0<\varepsilon<\rho-\eta}$ is non-decreasing i.e. if $0 < \varepsilon_1 \leq \varepsilon_2 < \rho - \eta$, then

$$\forall\, x \in \Omega_\rho, \quad [v *_k \varphi_\eta] *_k \varphi_{\varepsilon_1}(x) \leq [v *_k \varphi_\eta] *_k \varphi_{\varepsilon_2}(x).$$

By (4.7) the previous inequality can be written

$$\forall\, x \in \Omega_\rho, \quad [v *_k \varphi_{\varepsilon_1}] *_k \varphi_\eta(x) \leq [v *_k \varphi_{\varepsilon_2}] *_k \varphi_\eta(x).$$

Finally, letting $\eta \longrightarrow 0$ and using the statement iv) of Proposition 4.3, we obtain

$$\forall\, x \in \Omega_\rho, \quad v *_k \varphi_{\varepsilon_1}(x) \leq v *_k \varphi_{\varepsilon_2}(x).$$

This proves the assertion a).
b) Let $0 < \rho < r_\Omega$. Since the sequence $(v *_k \varphi_\varepsilon)_{0<\varepsilon<\rho}$ is non-decreasing on Ω_ρ, we deduce that for any $x \in \Omega_\rho$, $s(x) := \lim_{\varepsilon\to 0} v *_k \varphi_\varepsilon(x)$ exists in $[-\infty, +\infty[$. On the other hand, from Proposition 4.3-iii), we see that $s = v$ almost everywhere on Ω_ρ. In particular $s \neq -\infty$ on each connected component of Ω_ρ. Consequently, by a) and Proposition 3.3 we deduce that $s \in \mathcal{SH}_k(\Omega_\rho)$ as a pointwise decreasing limit of D-subharmonic functions on Ω_ρ. As $\rho > 0$ can be taken arbitrary small, the proof of the lemma is complete. □

Now, we will extend the results of Proposition 4.2 to any D-subharmonic function (see [1], Corollary 3.2.6 for the classical case).

Theorem 4.2 *Let u be an u.s.c. function on a W-invariant open set $\Omega \subset \mathbb{R}^d$. Assume that u is not identically $-\infty$ on each connected component of Ω. Then the statements i), ii), iii) and iv) of Proposition 4.2 are equivalent.*

Proof: **i) $\Longrightarrow$ ii)** Let $u \in \mathcal{SH}_k(\Omega)$. We already know that (4.1) holds (see Proposition 3.2). Let (u_ε) be the sequence defined by (4.4). By Theorem 4.1, $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon) \cap \mathcal{SH}_k(\Omega_\varepsilon)$. Therefore, using Proposition 4.2, $r \longmapsto M^r_B(u_\varepsilon)(x)$ is non-decreasing on $]0, dist(x, \partial\Omega_\varepsilon)[$.

Letting $\varepsilon \longrightarrow 0$ and using Theorem 4.1, 3), we deduce that $r \longmapsto M_B^r(u)(x)$ is also non-decreasing.
ii) $\Longrightarrow$ **i)** This is obvious.
i) $\Longrightarrow$ **iii)** If $u \in \mathcal{SH}_k(\Omega) \cap \mathcal{C}^\infty(\Omega)$, the result is proved in Proposition 4.2.
Let us now suppose only that $u \in \mathcal{SH}_k(\Omega)$. By Proposition 4.3 and Theorem 4.1, the functions u_ε defined by (4.4) are in $\mathcal{SH}_k(\Omega_\varepsilon) \cap \mathcal{C}^\infty(\Omega_\varepsilon)$.
Consequently, we have
a) the function $r \longmapsto M_S^r(u_\varepsilon)(x)$ is non-decreasing on $]0, dist(x, \partial\Omega_\varepsilon)[$,
b) for all $0 < \varepsilon < \rho$, $\lim_{r\to 0} M_S^r(u_\varepsilon)(x) = u_\varepsilon(x)$,
c) for all $0 < \varepsilon < \rho$, $u_\varepsilon(x) \leq M_S^r(u_\varepsilon)(x)$,
where $\rho = \rho(x) > 0$ is such that $x \in \Omega_\varepsilon$ for all $\varepsilon < \rho$.
From a) and Theorem 4.1-3), we can see that $r \longmapsto M_S^r(u)(x)$ is also non-decreasing as a pointwise limit of non-decreasing functions.
Using c) and letting $\varepsilon \longrightarrow 0$, we have $u(x) \leq M_S^r(u)(x)$. Moreover, since $(u_\varepsilon)_{0<\varepsilon<\rho}$ is a non-decreasing sequence, we deduce that

$$\forall\ 0 < \varepsilon < \rho, \quad u(x) \leq M_S^r(u)(x) \leq M_S^r(u_\varepsilon)(x).$$

According to b), this implies that

$$\forall\ 0 < \varepsilon < \rho, \quad u(x) \leq \lim_{r\to 0} M_S^r(u)(x) \leq \lim_{r\to 0} M_S^r(u_\varepsilon)(x) = u_\varepsilon(x).$$

Finally, letting $\varepsilon \longrightarrow 0$ and using Theorem 4.1-2), we deduce the desired result.
iii) $\Longrightarrow$ **i)** Let $x \in \Omega$ and $r \in]0, \varrho_x[$ be fixed and assume that u is nonpositive on the compact set $B^W(x, r)$ (using the upper semi-continuity of u). For all $\rho \in]0, r[$, we have

$$\frac{2\gamma + d}{r^{2\gamma+d}} \int_\rho^r M_S^t(u)(x) t^{2\gamma+d-1} dt \geq M_S^\rho(u)(x)\big(1 - (\rho/r)^{d+2\gamma}\big).$$

Since $t \longmapsto M_S^t(u)(x)$ is nonpositive on $]0, r]$, letting $\rho \longrightarrow 0$ and using the monotone convergence theorem, Lemma 2.1 and the relation (4.2), we obtain

$$M_B^r(u)(x) \geq u(x).$$

This proves that u is D-subharmonic on Ω.
i) $\Longrightarrow$ **iv)** Let $u \in \mathcal{SH}_k(\Omega)$. We know that the function $u\omega_k$ is locally integrable on Ω and $\lim_{r\to 0} M_B^r(u)(x) = u(x)$ for every $x \in \Omega$. By Proposition 4.2, the result is true when $u \in \mathcal{C}^2(\Omega)$. Now, suppose only that u is in $\mathcal{SH}_k(\Omega)$. Considering the D-subharmonic functions u_ε defined in Theorem 4.1 , we get for ε small enough

$$M_B^r(u_\varepsilon)(x) \leq M_S^r(u_\varepsilon)(x).$$

By Theorem 4.1, we deduce that $M_B^r(u)(x) \leq M_S^r(u)(x)$.
iv) $\Longrightarrow$ **i)** We will use the same idea as in [12] (Lemma 2.4.4). First, we need the following lemma:

Lemma 4.2 *Let $f \in L^1_{k,loc}(\Omega)$ be an u.s.c. function. Then for every $x \in \Omega$ and $r > 0$ such that $B(x,r) \subset \Omega$, the function $t \longmapsto M_S^t(f)(x)t^{d+2\gamma-1}$ is integrable on $[0,r]$ and we have*

$$M_B^r(f)(x) = \frac{d+2\gamma}{r^{d+2\gamma}} \int_0^r M_S^t(f)(x)t^{d+2\gamma-1}dt. \tag{4.10}$$

Proof: Assume that f is nonpositive in the fixed Dunkl ball $B^W(x,r) \subset \Omega$. The formula (4.10) has been established in Lemma 2.1. Therefore, it suffices to show that $M_B^r(f)(x) \neq -\infty$. Denoting $C_r := (m_k(B(0,r)))^{-1}$, by (2.1) and the fact that $h_k(r,x,y) \leq 1$, we get

$$|M_B^r(f)(x)| \leq C_r \int_{B^W(x,r)} |f(y)|h_k(r,x,y)\omega_k(y)dy \leq C_r \int_{B^W(x,r)} |f(y)|\omega_k(y)dy < +\infty.$$

$\square$

Now, we turn to the proof of $iv) \implies i)$. Let $x \in \Omega$. Suppose that $M_B^r(u)(x) \leq M_S^r(u)(x)$ for every $r \in]0, \varrho_x[$. Since $u \in L^1_{k,loc}(\Omega)$, by Lemma 4.2, the function $r \longmapsto M_B^r(u)(x)$ is absolutely continuous on every closed interval $[a,b] \subset]0, \varrho_x[$ as a product of two absolutely continuous functions. Hence, it is almost everywhere differentiable on $[a,b]$ and we have

$$\frac{d}{dr} M_B^r(u)(x) = \frac{d+2\gamma}{r}\big(M_S^r(u)(x) - M_B^r(u)(x)\big) \geq 0 \quad a.e..$$

Thus, $r \longmapsto M_B^r(u)(x)$ is non-decreasing on $[a,b]$ (see [2], Proposition 5.3). That is, for every $0 < t \leq r < \varrho_x$, we have $M_B^t(u)(x) \leq M_B^r(u)(x)$. Letting $t \longrightarrow 0$, we deduce that $u(x) \leq M_B^r(u)(x)$. This proves that u is in $\mathcal{SH}_k(\Omega)$ and the Theorem is completely proved.

$\square$

5 Δ_k-Riesz measure

In this section, we introduce the Riesz measure of a function $u \in \mathcal{SH}_k(\Omega)$. In order to do this, we will clarify some facts about the action of Dunkl operators on distributions. Let us start by recalling the following integration by parts formula see [5] or [23]): Let $f, g \in C^1(\Omega)$ such that g has compact support and D_ξ be the ξ-directional Dunkl operator defined by (1.1). Then we have

$$\int_\Omega D_\xi f(x)g(x)\omega_k(x)dx = -\int_\Omega f(x)D_\xi g(x)\omega_k(x)dx. \tag{5.1}$$

For a distribution $T \in \mathcal{D}'(\Omega)$, we define the weak Dunkl ξ-directional derivative of T ($\xi \in \mathbb{R}^d$) by

$$\forall\ \phi \in \mathcal{D}(\Omega), \quad \langle D_\xi T, \phi \rangle = -\langle T, D_\xi \phi \rangle\,.$$

Note that by the intertwining relation (1.2), the operator $D_\xi = V_k \partial_\xi V_k^{-1} : \mathcal{C}^\infty(\mathbb{R}^d) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^d)$ is continuous for the Fréchet topology. Moreover, since D_ξ leaves the space $\mathcal{D}(\Omega)$ invariant, we deduce that $D_\xi : \mathcal{D}(\Omega) \longrightarrow \mathcal{D}(\Omega)$ is also continuous for the Fréchet topology.

This justifies that $D_\xi T$ is well defined as an element of $\mathcal{D}'(\Omega)$. In particular, if $f \in L^1_{k,loc}(\Omega)$ i.e. $f\omega_k \in L^1_{loc}(\Omega)$, the weak Dunkl-Laplacian of $f\omega_k$ is given by

$$\forall\ \phi \in \mathcal{D}(\Omega), \quad \langle \Delta_k(f\omega_k), \phi\rangle = \langle f\omega_k, \Delta_k\phi\rangle = \int_\Omega f(x)\Delta_k\phi(x)\omega_k(x)dx. \tag{5.2}$$

Our first main result states that:

Theorem 5.1 *Let $u \in \mathcal{SH}_k(\Omega)$. Then there exists a nonnegative Radon measure μ in Ω such that $\Delta_k[u\omega_k] = \mu$ in the sense of distributions. We will call μ the Δ_k-Riesz measure related to u.*

Proof: As $u \in L^1_{k,loc}(\Omega)$, $u\omega_k$ defines a distribution. Let $\phi \in \mathcal{D}(\Omega)$ and let $(u_\varepsilon)_{0<\varepsilon<\rho}$ be the sequence of functions defined by (4.4) with ρ such that supp $\phi \subset \Omega_\rho$. As $0 \le u_\varepsilon - u \le u_\rho - u$, by Theorem 4.1 and the dominated convergence theorem, we have

$$\langle \Delta_k[u\omega_k], \phi\rangle = \int_\Omega u(x)\Delta_k\phi(x)\omega_k(x)dx = \lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\Delta_k\phi(x)\omega_k(x)dx.$$

Now, using the integration by parts formula (5.1), we deduce that

$$\langle \Delta_k[u\omega_k], \phi\rangle = \lim_{\varepsilon\to 0}\int_\Omega \Delta_k u_\varepsilon(x)\phi(x)\omega_k(x)dx. \tag{5.3}$$

Consequently, $[\Delta_k u_\varepsilon]\omega_k \longrightarrow \Delta_k[u\omega_k]$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \to 0$. Moreover, from (5.3) and the fact that $\Delta_k u_\varepsilon \ge 0$ (Theorem 4.1 and Proposition 4.1), we see that $\Delta_k[u\omega_k]$ is a nonnegative distribution on Ω. Then, according to [25], there exists a nonnegative Radon measure μ on Ω such that $\Delta_k[u\omega_k] = \mu$ and the proposition is proved. □

Example 5.1 *Let $u \in \mathcal{SH}_k(\Omega) \cap \mathcal{C}^2(\Omega)$. Using (5.2) and (5.1), clearly the Δ_k-Riesz measure of u is given by $\Delta_k u(x)\omega_k(x)dx$.*

Now, we will establish a type Weyl's lemma for D-subharmonic functions:

Theorem 5.2 *Let $u \in L^1_{k,loc}(\Omega)$. If $\Delta_k(u\omega_k) \ge 0$ in $\mathcal{D}'(\Omega)$, then there exists a D-subharmonic function s on Ω such that $u = s$ a.e. in Ω.*

Proof: Let us denote by μ the nonnegative Radon measure $\Delta_k(u\omega_k)$ and let φ_ε be the function given by (4.3). We claim that

$$\forall\ \varepsilon < r_\Omega, \quad \forall\ x \in \Omega_\varepsilon, \quad \Delta_k(u *_k \varphi_\varepsilon)(x) = \mu *_k \varphi_\varepsilon(x) := \int_\Omega \tau_{-x}\varphi_\varepsilon(y)d\mu(y)^8. \tag{5.4}$$

Indeed, by Proposition 4.3, the function $u *_k \varphi_\varepsilon$ is of class C^∞ on Ω_ε. Then, using respectively the relations (4.5), (4.4) and $(A.6)$, we get

$$\begin{aligned}\Delta_k(u *_k \varphi_\varepsilon)(x) &= [u *_k (\Delta_k\varphi_\varepsilon)](x) = \int_\Omega u(y)\tau_{-x}[\Delta_k\varphi_\varepsilon](y)\omega_k(y)dy \\ &= \int_\Omega u(y)\Delta_k[\tau_{-x}\varphi_\varepsilon](y)\omega_k(y)dy = \langle u\omega_k, \Delta_k[\tau_{-x}\varphi_\varepsilon]\rangle \\ &= \mu *_k \varphi_\varepsilon(x).\end{aligned}$$

[8]Note that by (4.8), $\mu *_k \varphi_\varepsilon$ is well defined on Ω_ε for any nonnegative Radon measure μ on Ω.

Since $\tau_{-x}\varphi_\varepsilon \geq 0$, (5.4) implies that $\Delta_k[u *_k \varphi_\varepsilon] \geq 0$ on Ω_ε. Hence, the function $u *_k \varphi_\varepsilon \in \mathcal{SH}_k(\Omega_\varepsilon)$ (see Proposition 4.1). Thus, we obtain the result by using Lemma 4.1, b). $\square$

In the following result, we characterize the D-subharmonicity by means of the positivity of the distributional Dunkl Laplacian.

Corollary 5.1 *Let u be a function defined on Ω. Then $u \in \mathcal{SH}_k(\Omega)$ if and only if u satisfies: $u \in L^1_{k,loc}(\Omega)$, $\Delta_k(u\omega_k) \geq 0$ in $\mathcal{D}'(\Omega)$ and $u(x) = \lim_{r\to 0} M_B^r(u)(x)$ for every $x \in \Omega$.*

Proof: The necessity part follows from Propositions 3.1, 3.2 and 5.1. Now, will show the sufficiency part. By Theorem 5.2, there exists a function $v \in \mathcal{SH}_k(\Omega)$ such that $u(x) = v(x)$ for almost every $x \in \Omega$. Therefore, for all $x \in \Omega$ and all $r > 0$ small enough, we have $M_B^r(u)(x) = M_B^r(v)(x)$. Now, using Proposition 3.2, we deduce that u and v are identically equal in Ω and then u is in $\mathcal{SH}_k(\Omega)$. $\square$

Corollary 5.2 *The cone $\mathcal{SH}_k(\Omega)$ is closed for the $L^1_{k,loc}(\Omega)$ topology.*

Proof: Let (u_n) be a sequence of D-subharmonic functions on Ω such that $u_n \longrightarrow u$ in $L^1_{k,loc}(\Omega)$. As, $u_n\omega_k$ and $u\omega_k$ are in $L^1_{loc}(\Omega)$, we deduce that $u_n\omega_k \longrightarrow u\omega_k$ in $\mathcal{D}'(\Omega)$. Hence, $\Delta_k(u_n\omega_k) \longrightarrow \Delta_k(u\omega)$ in $\mathcal{D}'(\Omega)$. By Corollary 5.1, as $\Delta_k(u_n\omega_k) \geq 0$, we deduce that $\Delta_k(u\omega_k) \geq 0$ in $\mathcal{D}'(\Omega)$. Now, by Theorem 5.2 there exists a D-subharmonic function s on Ω such that $u = s$ a.e. in Ω. Then $u = s$ in $L^1_{k,loc}(\Omega)$ and the result is proved. $\square$

In [10], Weyl's lemma for D-harmonic functions has been proved. Here, we will give another proof of such result. In order to do this, we will prove the following lemma:

Lemma 5.1 *A function $u : \Omega \longmapsto \mathbb{R}$ is D-harmonic if and only if it is simultaneously D-subharmonic and D-superharmonic on Ω.*

Proof: It is enough to show the sufficiency part. Let $\rho > 0$ small enough and consider the function u_ε, with $\varepsilon < \rho$, defined by (4.4). Clearly, by Theorem 4.1, the functions u_ε and $-u_\varepsilon$ are in $\mathcal{C}^\infty(\Omega_\rho) \cap \mathcal{SH}_k(\Omega_\rho)$. Hence, by Proposition 4.1, we deduce that u_ε is D-harmonic in Ω_ρ. Again from Proposition 4.1, $u_\varepsilon(x) = M_S^r(u_\varepsilon)(x)$ whenever $B(x,r) \subset \Omega_\rho$. Letting $\varepsilon \longrightarrow 0$ and using Theorem 4.1, we deduce that $u(x) = M_S^r(u)(x)$ whenever $B(x,r) \subset \Omega_\rho$. Since ρ is arbitrary small, we deduce that

$$u(x) = M_S^r(u)(x), \quad \text{for every} \quad B(x,r) \subset \Omega.$$

Finally, if we use (4.9), we conclude that for any $\varepsilon > 0$, u coincides with the D-harmonic function u_ε on Ω_ε. That is the function u is D-harmonic on Ω as desired. $\square$

Corollary 5.3 *If $u \in L^1_{k,loc}(\Omega)$ satisfies $\Delta_k[u\omega_k] = 0$ in $\mathcal{D}'(\Omega)$, then there exists a D-harmonic function h on Ω such that u and h coincide a.e. on Ω.*

Proof: From Theorem 5.2, there exist two functions u_1, u_2 such that u_1 is D-subharmonic on Ω, u_2 is D-superharmonic on Ω and $u = u_1 = u_2$ almost everywhere. Moreover, by Proposition 3.2, we have

$$\forall\ x \in \Omega, \quad u_1(x) = \lim_{r\to 0} M_B^r(u_1)(x) = \lim_{r\to 0} M_B^r(u_2)(x) = u_2(x).$$

Therefore, the function $h := u_1 = u_2$ is simultaneously D-subharmonic and D-superharmonic on Ω. Hence, by the first step, h is D-harmonic in Ω and $h = u$ almost everywhere in Ω. $\square$

6 Dunkl-Newtonian Potentials

In this section, we introduce the Dunkl-Newton kernel and the corresponding Dunkl-Newtonian potentials and we study some of their properties. Throughout this section, we will always suppose that $d+2\gamma>2$ (transient condition).

6.1 Dunkl type Newton kernel

Consider the Dunkl-Newton kernel defined by (1.14). It takes also the following form:

Proposition 6.1 *For every $x, y \in \mathbb{R}^d$, we have*

$$N_k(x,y) = \frac{1}{d_k(d+2\gamma-2)} \int_{\mathbb{R}^d} \Big(\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle \Big)^{\frac{2-(d+2\gamma)}{2}} d\mu_y(z). \tag{6.1}$$

Proof: From (1.7) and (1.10), we have

$$p_t(x,y) = \frac{1}{(2t)^{\frac{d}{2}+\gamma} c_k} \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2+\|y\|^2-2\langle x,z\rangle}{4t}} d\mu_y(z). \tag{6.2}$$

Hence, by the change of variables $1/4t \leftrightarrow t$ in the integral (1.14) and using (2.5), we can write

$$N_k(x,y) = \frac{1}{2d_k\Gamma(d/2+\gamma)} \int_0^{+\infty} t^{\frac{d}{2}+\gamma-2} \int_{\mathbb{R}^d} e^{-t(\|x\|^2+\|y\|^2-2\langle x,z\rangle)} d\mu_y(z)dt.$$

Applying Fubini's theorem and then using the identity $A^{-\lambda} = \frac{1}{\Gamma(\lambda)}\int_0^{+\infty} s^{\lambda-1}e^{-sA}ds$, $A \geq 0$ and $\lambda > 0$ (when $A = 0$, the both terms are equal to $+\infty$) by taking $A = \|x\|^2 + \|y\|^2 - 2\langle x, z\rangle$ and $\lambda = \frac{d+2\gamma-2}{2}$, we obtain the result. $\square$

Example 6.1 ***1)*** *When $k = 0$ and $d > 2$, the Rösler measure μ_x is equal to δ_x (the Dirac measure at x) and then $N_0(x,y) = \frac{1}{(d-2)\omega_{d-1}}\|x-y\|^{2-d}$ is the classical Newton kernel* [9].

2) *We consider $\mathbb{R}^d$ ($d \geq 1$) with the root system $R_m := \{\pm e_1, \ldots, \pm e_m\}$, where m is a fixed integer in $\{1, \ldots, d\}$ and $(e_j)_{1\leq j\leq d}$ is the canonical basis of $\mathbb{R}^d$. For $\xi \in \mathbb{R}^d$, we will denote $\xi = (\xi^{(m)}, \xi') \in \mathbb{R}^m \times \mathbb{R}^{d-m}$.*
Noting that the Coxeter-Weyl group is given by $W = \mathbb{Z}_2^m$ and that the $\mathbb{Z}_2^m$-orbit of a point $\xi \in \mathbb{R}^d$ is given by

$$\mathbb{Z}_2^m.\xi := \{\varepsilon.\xi := (\varepsilon_1\xi_1, \ldots, \varepsilon_m\xi_m, \xi'), \quad \varepsilon = (\varepsilon_i)_{1\leq i\leq m} \in \{\pm 1\}^m\}.$$

The multiplicity function can be represented by the m-multidimensional parameter $k = (k_1, \ldots, k_m)$ with $k_j = k(e_j) > 0$. Moreover, the Rösler measure is of the form $\mu_y = \mu_{(y^{(m)},y')} = \mu_{y_1} \otimes \cdots \otimes \mu_{y_m} \otimes \delta_{y'}$ with μ_{y_i} the $\mathbb{Z}_2$-Rösler measure at point y_i. If $y_i = 0$, we know that $\mu_0 = \delta_0$ and if $y_i \neq 0$, we have

$$\langle \mu_{y_i}, f\rangle := \int_{-1}^1 f(ty_i)\phi_{k_i}(t)dt, \quad f \in \mathcal{C}(\mathbb{R}),$$

[9] ω_{d-1} is the area of S^{d-1}.

where ϕ_{k_i} is the $\mathbb{Z}_2$-Dunkl density function of parameter k_i given by (see [4] or [23] p.104)

$$\phi_{k_i}(t) := \frac{\Gamma(k_i + 1/2)}{\sqrt{\pi}\Gamma(k_i)}(1-t)^{k_i-1}(1+t)^{k_i}\mathbf{1}_{[-1,1]}(t).$$

Let $C := [d_k(d + 2\gamma - 2)]^{-1}$. Then the $\mathbb{Z}_2^m$-Dunkl-Newton kernel is of the form

$$N_k^{\mathbb{Z}_2^m}(x,y) = C\int_{[-1,1]^m}\Big(\|x^{(m)}\|^2 + \|y^{(m)}\|^2 - 2\sum_{j=1}^m t_j x_j y_j + \|x' - y'\|^2\Big)^{1-\frac{d}{2}-\gamma}$$
$$\times \prod_{i=1}^m \phi_{k_i}(t_i)dt_1\dots dt_m.$$

Proposition 6.2 *Let $x, y \in \mathbb{R}^d$, with $x \neq 0$.*

1) *If $y \notin W.x$, then $0 < N_k(x,y) < +\infty$.*

2) *When $d \geq 2$ and $\gamma > 0$, we have $N_k(x,x) = +\infty$.*

Proof: **1)** Let $y \in \mathbb{R}^d$ fixed. It is well known (see [21] and [23]) that

$$p_t(x,y) \leq \frac{1}{(2t)^{\frac{d}{2}+\gamma}c_k}\max_{g\in W} e^{-(\|x-gy\|^2)/4t}. \tag{6.3}$$

Hence, $N_k(x,y) < +\infty$ for all $x \notin W.y$.
2) At first suppose that x is not in the hyperplanes H_α, $\alpha \in R$ (i.e. x lives in a Weyl chamber). It is enough to prove that $I := \int_0^1 p_t(x,x) = +\infty$. To do this, we need the following short-time asymptotic result of the Dunkl heat kernel established in [15] (Corollary 2): Let C be a fixed Weyl chamber. If $x, y \in C$, then

$$p_t(x,y) \sim_{t\to 0} \big(\omega_k(x)\omega_k(y)\big)^{-1/2}(4\pi t)^{-d/2}e^{-\frac{\|x-y\|^2}{4t}}.$$

For $y = x$, we obtain $p_t(x,x) \sim_{t\to 0} \big(\omega_k(x)\big)^{-1}(4\pi t)^{-d/2}$ and $I = +\infty$ as desired.
When $x \in H_\alpha$ for some $\alpha \in R$, the result follows by using the lower semi-continuity of the function $x \longmapsto N_k(x,x)$ (as non-decreasing limit of the sequence of continuous functions $x \longmapsto \int_{1/n}^n p_t(x,x)dt$). Indeed, if $x \in H_\alpha$, $N_k(x,x) = \liminf_{y\to x} N_k(y,y) = +\infty$ because $N_k(y,y) = +\infty$ if y converges to x in a Weyl chamber limited by H_α. $\square$

Remark 6.1 *For $g \neq id$, it is much more difficult to see if $N_k(x, gx)$ is finite or infinite. This will be more explained in a forthcoming paper. However, from the relation (6.5) (see the next result), we can see that $N_k(x, gx) = +\infty$ if and only if $N_k(x, g^{-1}x) = +\infty$.*

Proposition 6.3 *The Dunkl-Newton kernel satisfies the following properties:*

1. *For all $x, y \in \mathbb{R}^d$, we have*

$$N_k(x,y) = \frac{1}{d_k}\int_0^{+\infty} t^{1-d-2\gamma}h_k(t,x,y)dt. \tag{6.4}$$

2. For every $x, y \in \mathbb{R}^d$. Then

$$N_k(x,y) = N_k(y,x), \quad N_k(gx,y) = N_k(x,g^{-1}y). \tag{6.5}$$

3. For all $x, y \in \mathbb{R}^d$ with $x \notin W.y$, we have

$$\min_{g\in W}\Big(\|x-gy\|^{2-(d+2\gamma)}\Big) \le d_k(d+2\gamma-2)N_k(x,y) \le \max_{g\in W}\Big(\|x-gy\|^{2-(d+2\gamma)}\Big). \tag{6.6}$$

4. For all $y \in \mathbb{R}^d$ fixed, the function $x \longmapsto N_k(x,y)$ is lower semi-continuous (l.s.c.) on $\mathbb{R}^d$ and of class C^∞ on $\mathbb{R}^d \backslash W.y$.

Proof: **1.** Fix $x, y \in \mathbb{R}^d$. By (6.1) and Fubini's theorem , we have

$$\begin{aligned} N_k(x,y) &= \frac{1}{d_k}\int_{\mathbb{R}^d}\Big(\int_{\sqrt{\|x\|^2+\|y\|^2-2\langle x,z\rangle}}^{+\infty} t^{1-(d+2\gamma)}dt\Big)d\mu_y(z) \\ &= \frac{1}{d_k}\int_0^{+\infty} t^{1-(d+2\gamma)}\Big(\int_{\mathbb{R}^d}\mathbf{1}_{[0,t]}(\sqrt{\|x\|^2+\|y\|^2-2\langle x,z\rangle})d\mu_y(z)\Big)\,dt \\ &= \frac{1}{d_k}\int_0^{+\infty} t^{1-(d+2\gamma)}h_k(t,x,y)dt. \end{aligned}$$

2. We obtain (6.5) by using (6.4) and the properties (2.3) of the harmonic kernel.
3. At first, we note that from (1.3) for $z \in$ supp μ_y we can write $z = \sum_{g\in W}\lambda_g(z)gy$, where $\lambda_g(z) \in [0,1]$ are such that $\sum_{g\in W}\lambda_g(z) = 1$. Then we have

$$\|x\|^2 + \|y\|^2 - 2\langle x,z\rangle = \sum_{g\in W}\lambda_g(z)\|x-gy\|^2. \tag{6.7}$$

Now, as $f : t \longmapsto t^{1-\frac{d}{2}-\gamma}$ is a convex function on $]0,+\infty[$, by (6.7) we have

$$\Big(\|x\|^2+\|y\|^2-2\langle x,z\rangle\Big)^{1-\frac{d}{2}-\gamma} \le \max_{g\in W}\big(\|x-gy\|^{2-(d+2\gamma)}\big).$$

This implies the right inequality. Again by convexity of the function f, Jensen's inequality and (6.7), we get

$$\begin{aligned} d_k(d+2\gamma-2)N_k(x,y) &\ge \Big(\int_{\mathbb{R}^d}(\|x\|^2+\|y\|^2-2\langle x,z\rangle)d\mu_y(z)\Big)^{\frac{2-(d+2\gamma)}{2}} \\ &= \Big(\sum_{g\in W}\big(\int_{\mathbb{R}^d}\lambda_g(z)d\mu_y(z)\big)\|x-gy\|^2\Big)^{\frac{2-(d+2\gamma)}{2}} \\ &\ge \Big(\max_{g\in W}\|x-gy\|^2\Big)^{\frac{2-(d+2\gamma)}{2}} = \min_{g\in W}\big(\|x-gy\|^{2-(d+2\gamma)}\big), \end{aligned}$$

where in the last line we have used the fact that f is a decreasing function.
4. The function $x \mapsto N_k(x,y)$ is l.s.c. on $\mathbb{R}^d$ as being the increasing limit of the sequence (f_n) of continuous functions defined by $f_n : x \mapsto \int_{1/n}^n p_t(x,y)dt$.
As μ_y is with compact support, we can differentiate locally in a neighborhood of $x \notin W.y$ under the integral in the relation (6.1) and we obtain the result. □

Theorem 6.1 *Let $x_0 \in \mathbb{R}^d$. Then the function $N_k(x_0, .)$ is*

1) *D-superharmonic on $\mathbb{R}^d$,*

2) *locally integrable on $\mathbb{R}^d$ with respect to the measure $\omega_k(x)dx$ and we have*

$$-\Delta_k\big(N_k(x_0, .)\omega_k\big) = \delta_{x_0} \quad in \quad \mathcal{D}'(\mathbb{R}^d), \tag{6.8}$$

where δ_{x_0} is the Dirac measure at x_0.

3) *D-harmonic on $\mathbb{R}^d \backslash W.x_0$ ($W.x_0$ the W-orbit of x_0).*

Proof: Fix $x_0 \in \mathbb{R}^d$. We will use the following properties of the Dunkl heat kernel (see [21])

$$(\Delta_k - \partial_t)\, p_t(x_0, .)(x) = 0 \quad \text{and} \quad \lim_{t\to 0} p_t(x_0, .)\omega_k = \delta_{x_0} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d), \tag{6.9}$$

We consider the function

$$S_{x_0,r}(x) := \int_r^{+\infty} p_t(x_0, x)dt. \tag{6.10}$$

1) By the monotone convergence theorem, we see that the function $N_k(x_0, .)$ is the pointwise increasing limit of the sequence $\big(S_{x_0,1/n}\big)_n$. Hence, by Proposition 3.3, it suffices to prove that for every $r > 0$, $S_{x_0,r}$ is D-superharmonic on $\mathbb{R}^d$. To do this, we will use the result of Proposition 4.1.
The function $p_t(x_0, .)$ is of class C^∞ on $\mathbb{R}^d$ and we can differentiate under the integral sign in the relation (6.2) to obtain

$$\partial_j p_t(x_0, .)(x) = -\frac{1}{2t}\frac{1}{(2t)^{\frac{d}{2}+\gamma}c_k}\int_{\mathbb{R}^d}(x_j - z_j)e^{-\frac{1}{4t}(\|x\|^2+\|x_0\|^2-2\langle x,z\rangle)}d\mu_{x_0}(z)$$

and

$$\begin{aligned}\partial_i\partial_j p_t(x_0, .)(x) = &-\delta_{ij}\frac{1}{2t}p_t(x_0, x)\\ &+\frac{1}{4t^2}\frac{1}{(2t)^{\frac{d}{2}+\gamma}c_k}\int_{\mathbb{R}^d}(x_j - z_j)(x_i - z_i)e^{-\frac{1}{4t}(\|x\|^2+\|x_0\|^2-2\langle x,z\rangle)}d\mu_{x_0}(z),\end{aligned}$$

where δ_{ij} is the Kronecker symbol.
Using the fact that supp $\mu_{x_0} \subset B(0, \|x_0\|)$, we deduce that

$$|\partial_j p_t(x_0, .)(x)| \leq \frac{\|x\| + \|x_0\|}{(2t)^{1+\frac{d}{2}+\gamma}c_k},$$

$$|\partial_i\partial_j p_t(x_0, .)(x)| \leq \frac{1}{(2t)^{1+\frac{d}{2}+\gamma}c_k} + \frac{(\|x\| + \|x_0\|)^2}{(2t)^{2+\frac{d}{2}+\gamma}c_k}.$$

Let $R > 0$. The previous inequalities and the differentiation theorem under the integral sign imply that $S_{x_0,r}$ is of class C^2 on the open ball $\overset{\circ}{B}(0,R)$ and by (6.9) we deduce for any $x \in \overset{\circ}{B}(0,R)$ that

$$\Delta_k S_{x_0,r}(x) = \int_r^{+\infty} \Delta_k \left(p_t(x_0,.)\right)(x)dt = \int_r^{+\infty} \partial_t p_t(x_0,x)dt = -p_r(x_0,x) < 0. \qquad (6.11)$$

Therefore, $S_{x_0,r}$ is D-superharmonic on $\overset{\circ}{B}(0,R)$. As $R > 0$ is arbitrary, we conclude that $S_{x_0,r}$ is D-superharmonic on $\mathbb{R}^d$ as desired.

2) From the statement 1) and Proposition 3.1, we deduce that $N_k(x_0,.) \in L^1_{k,loc}(\mathbb{R}^d)$. By the dominated convergence theorem, we can see that $S_{x_0,r}\omega_k \longrightarrow N_k(x_0,.)\omega_k$ in $\mathcal{D}'(\mathbb{R}^d)$ as $r \longrightarrow 0$. This implies that

$$\Delta_k(S_{x_0,r}\omega_k) \longrightarrow \Delta_k(N_k(x_0,.)\omega_k) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d) \quad \text{as} \quad r \longrightarrow 0.$$

On the other hand, from (6.11), (5.1) and (6.9), we have

$$\lim_{r\to 0} \Delta_k(S_{x_0,r}\omega_k) = -\delta_{x_0} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d).$$

This gives (6.8).

3) From the relation (6.8), we deduce that the function $N_k(x_0,.)\omega_k$ is D-harmonic in the sense of distributions on $\mathbb{R}^d\backslash\{x_0\}$. Hence, by applying Weyl's Lemma (see Corollary 5.3) on the W-invariant open set $\mathbb{R}^d\backslash W.x_0$, there exists a D-harmonic function h on $\mathbb{R}^d\backslash W.x_0$ such that $N_k(x_0,x) = h(x)$ for almost every $x \in \mathbb{R}^d\backslash W.x_0$. Now, using the smoothness of the function $N_k(x_0,.)$ on $\mathbb{R}^d\backslash W.x_0$, we obtain $N_k(x_0,.) = h$ on $\mathbb{R}^d\backslash W.x_0$.
This completes the proof. □

6.2 Dunkl-Newtonian potential of Radon measures

Definition 6.1 *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$. The Dunkl-Newtonian potential of μ is defined by*

$$N_k[\mu](x) := \int_{\mathbb{R}^d} N_k(x,y)d\mu(y), \quad x \in \mathbb{R}^d. \qquad (6.12)$$

Remark 6.2 *Let μ be a signed Radon measure on $\mathbb{R}^d$ and $\mu = \mu^+ - \mu^-$ its Hahn-Jordan decomposition. We can also define the Dunkl-Newtonian potential of μ by setting $N_k[\mu](x) := N_k[\mu^+](x) - N_k[\mu^-](x)$ whenever for every $x \in \mathbb{R}^d$, $N_k[\mu^+](x)$ and $N_k[\mu^-](x)$ are not infinite simultaneously.*

Proposition 6.4 *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$. A necessary and sufficient condition for finiteness a.e. of the Dunkl-Newtonian potential of μ is that*

$$\int_{\mathbb{R}^d} (1 + \|y\|)^{2-d-2\gamma}\, d\mu(y) < +\infty. \qquad (6.13)$$

We need the following lemma:

Lemma 6.1 *Let μ be a finite nonnegative Radon measure on $\mathbb{R}^d$. Then $N_k[\mu]$ belongs to $L^1_{k,loc}(\mathbb{R}^d)$. In particular, $N_k[\mu]$ is finite a.e..*

Proof: Fix $R > 0$. Using Fubini's theorem, we have

$$\int_{B(0,R)} N_k[\mu](x)\omega_k(x)dx = \int_{\mathbb{R}^d}\int_{B(0,R)} N_k(x,y)\omega_k(x)dx \; d\mu(y).$$

As $\mu(\mathbb{R}^d) < +\infty$, it suffices to show that there exists a constant $C = C(R, d, \gamma) > 0$ such that

$$\forall \; y \in \mathbb{R}^d, \quad \int_{B(0,R)} N_k(x,y)\omega_k(x)dx \leq C. \tag{6.14}$$

Let $x \in B(0,R)$ and $y \in \mathbb{R}^d$. From the relations (6.4), we can write

$$N_k(x,y) = \frac{1}{d_k}\int_0^1 t^{1-d-2\gamma}h_k(t,x,y)dt + \frac{1}{d_k}\int_1^{+\infty} t^{1-d-2\gamma}h_k(t,x,y)dt := I(x,y) + J(x,y).$$

- Since $h_k(t,x,y) \leq 1$, we can see that $J \leq \frac{1}{d_k(d+2\gamma-2)}$. This implies that

$$\forall \; y \in \mathbb{R}^d, \quad \int_{B(0,R)} J(x,y)\omega_k(x)dx \leq \frac{m_k[B(0,R)]}{d_k(d+2\gamma-2)} = C_1.$$

- Applying Fubini's theorem and then using (2.3) and (2.4), we deduce that

$$\begin{aligned}\forall \; y \in \mathbb{R}^d, \quad \int_{B(0,R)} I(x,y)\omega_k(x)dx &\leq \frac{1}{d_k}\int_0^1 t^{1-d-2\gamma}\|h_k(t,y,.)\|_{L^1(\mathbb{R}^d,m_k)}dt \\ &= \frac{1}{2(d+2\gamma)} = C_2.\end{aligned}$$

Finally, we obtain (6.14) by taking $C = C_1 + C_2$. □

Proof of Proposition 6.4. Assume that (6.13) holds. We will show that $x \longmapsto N_k[\mu](x)\omega_k(x)$ is locally integrable. Let $r \geq 1$. By Fubini's theorem, we have

$$\begin{aligned}\int_{B(0,r)} N_k[\mu](x)\omega_k(x)dx &= \int_{\|y\|\leq 2r}\Big(\int_{B(0,r)} N_k(x,y)\omega_k(x)dx\Big)d\mu(y) \\ &+ \int_{\|y\|>2r}\Big(\int_{B(0,r)} N_k(x,y)\omega_k(x)dx\Big)d\mu(y) = J_1 + J_2.\end{aligned}$$

From Lemma 6.1, $J_1 < +\infty$. Now, by (6.6), we have

$$J_2 \leq \frac{1}{d_k(d+2\gamma-2)}\int_{\|y\|>2r}\Big(\int_{B(0,r)} \max_{g\in W}\Big(\|x-gy\|^{2-d-2\gamma}\Big)\,\omega_k(x)dx\Big)d\mu(y).$$

But, for all $x \in B(0,r)$ and all $g \in W$, $\|x - gy\| \geq \|y\| - \|x\| \geq \frac{1}{2}\|y\|$ because $\|y\| \geq 2r$. Moreover, since $r \geq 1$, we also have $\|y\| \geq \frac{1}{2}(1 + \|y\|)$. Hence, we get

$$\forall\, g \in W, \quad \|x - gy\| \geq \frac{1}{4}(1 + \|y\|).$$

Thus,

$$J_2 \leq \frac{4^{d+2\gamma-2} m_k[B(0,r)]}{d_k(d+2\gamma-2)} \int_{\|y\|>2r} (1 + \|y\|)^{2-d-2\gamma} d\mu(y) < +\infty.$$

Conversely, suppose that (6.13) does not hold. Let $x \in B(0,1)$. Using (6.6) and the inequality $\|x - gy\| \leq 1 + \|y\|$ for all $g \in W$, we deduce that

$$\begin{aligned} d_k(d+2\gamma-2)N_k[\mu](x) &= d_k(d+2\gamma-2)\int_{\mathbb{R}^d} N_k(x,y)d\mu(y) \\ &\geq \int_{\mathbb{R}^d} \Big(\max_{g\in W}\|x-gy\|\Big)^{2-(d+2\gamma)} d\mu(y) \\ &\geq \int_{\mathbb{R}^d} \Big(1+\|y\|\Big)^{2-(d+2\gamma)} d\mu(y). \end{aligned}$$

Hence, if $\int_{\mathbb{R}^d} \big(1 + \|y\|\big)^{2-(d+2\gamma)} d\mu(y) = +\infty$, then $N_k[\mu](x) = +\infty$ on $B(0,1)$ and we get a contradiction. $\square$

Proposition 6.5 *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ with compact support. Then*

$$N_k[\mu](x) \sim \frac{\mu(\mathbb{R}^d)}{d_k(d+2\gamma-2)} \|x\|^{2-(d+2\gamma)} \quad as \quad \|x\| \longrightarrow +\infty.$$

Proof: Let $R > 0$ such that supp $\mu \subset B(0,R)$. By the Cauchy-Schwarz inequality, we have

$$\forall\, z \in \text{supp}\ \mu_y \subset B(0,\|y\|), \quad (\|x\| - \|y\|)^2 \leq \|x\|^2 + \|y\|^2 - 2\langle x, z\rangle \leq (\|x\| + \|y\|)^2.$$

Therefore, by (6.1) we obtain for every $y \in B(0,R)$ fixed and $\|x\| \geq 2R$

$$(\|x\| + \|y\|)^{2-d-2\gamma} \leq C.N_k(x,y) \leq (\|x\| - \|y\|)^{2-d-2\gamma},$$

where $C = d_k(d+2\gamma-2)$. If we integrate these inequalities with respect to the measure $d\mu(y)$ and we divide by $\|x\|^{2-d-2\gamma}$, we obtain the result by letting $\|x\| \longrightarrow +\infty$. $\square$

Proposition 6.6 *Let μ be a nonnegative Radon measure on $\mathbb{R}^d$.*

i) *If μ has compact support, then $N_k[\mu]$ is D-superharmonic on $\mathbb{R}^d$ and D-harmonic on $\mathbb{R}^d \backslash W.supp\ \mu$.*

ii) *If $N_k[\mu](x) < +\infty$ for at least one x, then $N_k[\mu]$ is D-superharmonic on $\mathbb{R}^d$.*

Proof: **i)** Let μ be a compactly supported and nonnegative Radon measure on $\mathbb{R}^d$.
• For $n \geq 1$, consider the function

$$F_n(x) := \int_{\text{supp }\mu} \Big(\int_{1/n}^{n} p_t(x,y)dt \Big) d\mu(y).$$

By the continuity theorem under the integral sign, we can see that F_n is continuous on $\mathbb{R}^d$. Furthermore, using the monotone convergence theorem, we deduce that $N_k[\mu]$ is a pointwise increasing limit of the sequence (F_n) of continuous functions. Therefore, the lower semi-continuity of the function $N_k[\mu]$ on $\mathbb{R}^d$ follows.
Let $x \in \mathbb{R}^d$ and $r > 0$. Using Fubini's theorem and the D-superharmonicity of the function $\xi \longmapsto N_k(\xi, y)$, we have

$$M_B^r(N_k[\mu])(x) = \int_{\mathbb{R}^d} M_B^r[N_k(.,y)](x)d\mu(y) \leq \int_{\mathbb{R}^d} N_k(x,y)d\mu(y) = N_k[\mu](x).$$

This implies that $N_k[\mu]$ is D-superharmonic on $\mathbb{R}^d$.
• According to Lemma 5.1, we need only to prove that $N_k[\mu]$ is D-subharmonic on $\Omega := \mathbb{R}^d \backslash W.\text{supp }\mu$. Let $B(x,r) \subset \Omega$. Again, by Fubini's theorem and the D-harmonicity of $N_k(.,y)$ on $\mathbb{R}^d \setminus W.y$, we deduce that

$$M_B^r(N_k[\mu])(x) = \int_{\mathbb{R}^d} M_B^r[N_k(.,y)](x)d\mu(y) = \int_{\mathbb{R}^d} N_k(x,y)d\mu(y) = N_k[\mu](x).$$

In particular, $N_k[\mu]$ satisfies the sub-mean property.
Now, it remains to show that $N_k[\mu]$ is u.s.c. on Ω. In fact, $N_k[\mu]$ is continuous on Ω. Indeed, fix $x_0 \in \Omega$ and $R > 0$ such that $\delta := dist\,(B(x_0,R), W.\text{supp }\mu) > 0$. We know that $x \mapsto N_k(x,y)$ is continuous on Ω for every $y \in \text{supp }\mu$. Moreover, from (6.3), we deduce that

$$\forall\ x \in B(x_0,R), \quad \forall\ y \in \text{supp }\mu, \quad p_t(x,y) \leq \frac{1}{(2t)^{\frac{d}{2}+\gamma} c_k} e^{-\delta/4t}.$$

This implies that

$$\forall\ (x,y) \in B(x_0,R) \times \text{supp }\mu, \quad N_k(x,y) \leq \int_0^{+\infty} \frac{1}{(2t)^{\frac{d}{2}+\gamma} c_k} e^{-\delta/4t} dt := C_\delta < +\infty.$$

Consequently, by the continuity theorem under the integral sign, we conclude that $N_k[\mu]$ is continuous on $B(x_0,R)$. This finishes the proof of i).

ii) Assume that $N_k[\mu](x_0) < +\infty$ for some $x_0 \in \mathbb{R}^d$. We consider the sequence of functions defined by

$$\phi_n(x) = \int_{B(0,n)} N_k(x,y)d\mu(y).$$

From i), we see that ϕ_n is D-superharmonic on $\mathbb{R}^d$ and $\phi_n(x) \uparrow N_k[\mu](x)$ as $n \longrightarrow +\infty$. Hence, from Proposition 3.3 the function $N_k[\mu]$ is D-superharmonic on $\mathbb{R}^d$. $\square$

Proposition 6.7 *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ satisfying the finiteness condition (6.13). Then $N_k[\mu]$ satisfies the Dunkl-Poisson equation*

$$-\Delta_k\big(N_k[\mu]\omega_k\big) = \mu \quad in\ \mathcal{D}'(\mathbb{R}^d). \tag{6.15}$$

Proof: By Proposition 6.6, $N_k[\mu]$ is D-superharmonic and then the function $N_k[\mu]\omega_k$ defines a distribution on $\mathbb{R}^d$. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Using the fact that $N_k[\mu]\omega_k$ is locally integrable, we can apply Fubini's theorem to obtain

$$\begin{aligned}\langle \Delta_k\big(N_k[\mu]\omega_k\big), \varphi\rangle &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} N_k(x,y)d\mu(y)\Big)\Delta_k\varphi(x)\omega_k(x)dx\\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} N_k(x,y)\Delta_k\varphi(x)\omega_k(x)dx\Big)d\mu(y)\\ &= \int_{\mathbb{R}^d}\langle \Delta_k\big(N_k(.,y)\omega_k\big), \varphi\rangle\, d\mu(y).\end{aligned}$$

As $N_k(x,y) = N_k(y,x)$, from (6.8) we obtain $\langle \Delta_k\big(N_k[\mu]\omega_k\big), \varphi\rangle = -\int_{\mathbb{R}^d}\varphi(y)d\mu(y)$, as desired. $\square$

From the previous result, we can deduce the uniqueness principle which states that

Corollary 6.1 *Let $\mu, \nu \in \mathcal{M}^+(\mathbb{R}^d)$. Assume that μ and ν satisfy (6.13) and $N_k[\mu] = N_k[\nu]$ a.e. on $\mathbb{R}^d$. Then $\mu = \nu$.*

In the following result, we will obtain all distributional solutions of the Dunkl-Poisson equation (see [16] for the classical case):

Proposition 6.8 *Let $f \in L^1_{loc}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}(1 + \|y\|)^{2-d-2\gamma}|f(y)|dy < +\infty$. Then the function $N_k[f] : x \mapsto \int_{\mathbb{R}^d} N_k(x,y)f(y)dy$ is a solution of the Poisson equation:*

$$-\Delta_k(u\omega_k) = f \quad in \quad \mathcal{D}'(\mathbb{R}^d). \tag{6.16}$$

Moreover, any solution u of (6.16) in $L^1_{k,loc}(\mathbb{R}^d)$ is of the form $N_k[f] + h$, where h is a D-harmonic function on $\mathbb{R}^d$.

Proof: By decomposing $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$, we may assume that f is nonnegative. Using Proposition 6.4, we deduce that $N_k[f]$ is finite a.e and Proposition 6.7 implies that it satisfies the Poisson equation (6.16).
Now, let v be a solution of (6.16). Then $\Delta_k(v\omega_k - N_k[f]\omega_k) = 0$ in distributional sense. Thus, by Weyl's lemma $v = N_k[f] + h$ a.e for some D-harmonic function h on $\mathbb{R}^d$. That is $v = N_k[f] + h$ in $L^1_{k,loc}(\mathbb{R}^d)$.

$\square$

7 Decompositions of Dunkl subharmonic functions

7.1 Riesz decomposition theorems

One of the most fundamental results in the theory of classical subharmonic functions is due to F. Riesz ([20]) and states that any subharmonic function can be locally written as the sum of a Newtonian potential plus a harmonic function (see for example [11]). In the following result, we will obtain an analog of this result for D-subharmonic functions.

Theorem 7.1 *Let $\Omega \subset \mathbb{R}^d$ be open and W-invaraint, $u \in \mathcal{SH}_k(\Omega)$ and $\mu = \Delta_k[u\omega_k]$ be the Δ_k-Riesz measure related to u. Then, for all W-invariant open set G with compact closure $\overline{G} \subset \Omega$, there exists a unique D-harmonic function h_G on G such that*

$$\forall\ x \in G, \quad u(x) = -\int_G N_k(x,y)d\mu(y) + h_G(x). \tag{7.1}$$

Proof: Let G be a W-invariant open set with compact closure $\overline{G} \subset \Omega$ and set $\mu_G := \mu_{|G}$ the restriction of μ to G. Clearly, μ_G is a nonnegative Radon measure on Ω with compact support contained in $\overline{G}$. It is also the Δ_k-Riesz measure of the restriction of u to G. Furthermore, μ_G can be considered as a compactly supported nonnegative Radon measure on $\mathbb{R}^d$. Hence, by Proposition 6.6, the function $N_k[\mu_G]$ is D-superharmonic on $\mathbb{R}^d$ (then also on G) and by the relation (6.15), we obtain

$$\Delta_k\left(u\omega_k + N_k[\mu_G]\omega_k\right) = 0 \quad \text{in} \quad \mathcal{D}'(G).$$

That is $u\omega_k + N_k[\mu_G]\omega_k$ is a D-harmonic distribution on G. By Weyl's lemma, there exists a D-harmonic function h_G on G such that $u(x) = -N_k[\mu_G](x) + h_G(x)$, for almost every $x \in G$. Finally, using the uniqueness principle (Corollary 3.1) we obtain the equality everywhere on G. □

Now, we will give a global version of the Riesz decomposition theorem:

Theorem 7.2 *Let Ω be a connected and W-invariant open subset of $\mathbb{R}^d$, $u \in \mathcal{SH}_k(\Omega)$ and let μ be the Δ_k-Riesz measure of u. Assume that $N_k[\mu](x) < +\infty$ for at least one $x \in \Omega$. Then there is a unique D-harmonic function h on Ω such that*

$$\forall\ x \in \Omega, \quad u(x) = -N_k[\mu](x) + h(x), \tag{7.2}$$

where $N_k[\mu](x) := \int_\Omega N_k(x,y)d\mu(y)$. In this case, we say that u has a global Riesz decomposition on Ω.

Proof: Let (O_n) be an open W-invariant exhaustion of Ω such that for every n (large enough) the compact closure of O_n is contained in O_{n+1} (we can take $O_n := \Omega_{\frac{1}{n}} \cap \overset{\circ}{B}(0,n)$, with Ω_r given by (2.10)) and let $\mu_n = \mu_{|O_n}$. As above, the function $N_k[\mu_n] : x \mapsto \int_{O_n} N_k(x,y)d\mu(y)$ is D-superhamonic on $\mathbb{R}^d$ and also on Ω.
Consequently, using the monotone convergence theorem, our hypothesis and Proposition 3.3, we deduce that $N_k[\mu]$ is D-superharmonic on Ω as being an increasing pointwise limit of a sequence of D-superharmonic functions on Ω. In particular, this implies that the function $N_k[\mu]\omega_k$ defines a distribution on Ω (by Proposition 3.1).
Now, if we use (6.8) and we proceed as in the proof of Proposition 6.7, we obtain

$$-\Delta_k\left(N_k[\mu]\omega_k\right) = \mu \quad \text{in} \quad \mathcal{D}'(\Omega). \tag{7.3}$$

Finally, we conclude the result by the same way, replacing G by Ω, as in the end of the proof of Theorem 7.1. □

Remark 7.1 *In the relation (7.1) (resp. (7.2) on Ω), we see that $h_G \geq u$ on G (resp. $h \geq u$). In this case, we say that h_G (resp. h) is a D-harmonic majorant of u on G (resp. on Ω). When $\Omega = \mathbb{R}^d$ and under the same assumptions of Theorem 7.2, we will prove in the next section that h is the least D-harmonic majorant of u on $\mathbb{R}^d$ in the sense that if h_1 is a D-harmonic function on $\mathbb{R}^d$, then $u \leq h_1$ implies $h \leq h_1$.*

7.2 Bounded from above Dunkl subharmonic functions on $\mathbb{R}^d$

In this subsection, we will describe the D-subharmonic functions which are bounded from above on the whole space $\mathbb{R}^d$ and we will characterize their related Riesz measures.

Theorem 7.3 *Let u be a bounded from above D-subharmonic function on $\mathbb{R}^d$ and μ be the associated Δ_k-Riesz measure. Then u has a global Riesz decomposition on $\mathbb{R}^d$ given by*

$$u(x) = \sup_{x\in\mathbb{R}^d} u(x) - N_k[\mu](x), \quad x \in \mathbb{R}^d. \tag{7.4}$$

In the classical case, the proof of this theorem is based on the Nivanlinna theorems (see [11], Theorem 3.20). Here, we will give another proof. We start by the following result:

Lemma 7.1 *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ and $\mu *_k \varphi_\varepsilon$, $\varepsilon > 0$, be the function defined on $\mathbb{R}^d$ by (5.4). Then, for every $x \in \mathbb{R}^d$, we have*

$$N_k\big[(\mu *_k \varphi_\varepsilon)(y)\omega_k(y)dy\big](x) = \int_{\mathbb{R}^d} N(x,.) *_k \varphi_\varepsilon(z)d\mu(z) \tag{7.5}$$

and

$$\lim_{\varepsilon\to 0} N_k\big[(\mu *_k \varphi_\varepsilon)(y)\omega_k(y)dy\big](x) = N_k[\mu](x). \tag{7.6}$$

Note that the terms in (7.5) and (7.6) may be equal to $+\infty$.

Proof: **i)** Let $x \in \mathbb{R}^d$ and $\varepsilon > 0$. We obtain (7.5) by using respectively (6.12), (5.4), Fubini's theorem and (A.8) as follows

$$\begin{aligned} N_k\big[(\mu *_k \varphi_\varepsilon)(y)\omega_k(y)dy\big](x) &= \int_{\mathbb{R}^d} N_k(x,y)\Big(\int_{\mathbb{R}^d} \tau_{-y}\varphi_\varepsilon(z)d\mu(z)\Big)\omega_k(y)dy \\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} N_k(x,y)\tau_{-z}\varphi_\varepsilon(y)\omega_k(y)dy\Big)d\mu(z) \\ &= \int_{\mathbb{R}^d} N(x,.) *_k \varphi_\varepsilon(z)d\mu(z). \end{aligned}$$

ii) As the function $N_k(x,.)$ is D-superharmonic on $\mathbb{R}^d$, by Theorem 4.1, $N(x,.)$ is the decreasing pointwise limit of the sequence $(N_k(x,.) *_k \varphi_\varepsilon)_\varepsilon$ as $\varepsilon \to 0$. Consequently, (7.6) follows from (7.5) and from the monotone convergence theorem. $\square$

Proof of Theorem 7.3: We shall prove first the result when u is of class C^2 on $\mathbb{R}^d$. In this case, the relation (2.16) plays a key role.
Let $a := \sup_{x\in\mathbb{R}^d} u(x)$. We can see by (2.14) that $M_S^r(u)(x) \leq a$ for every $x \in \mathbb{R}^d$ and

every $r > 0$. Moreover, since $u \in \mathcal{SH}_k(\mathbb{R}^d)$, the function $r \longmapsto M_S^r(u)(x)$ is non decreasing (by Proposition 4.2). Consequently, $h(x) := \lim_{r\to+\infty} M_S^r(u)(x)$ exists and $h(x) \leq a$ for every $x \in \mathbb{R}^d$.
On the other hand, as $\Delta_k u \geq 0$, by the monotone convergence theorem, we have

$$\lim_{r\to+\infty} \frac{1}{d+2\gamma}\int_0^r M_B^t(\Delta_k u)(x) t\, dt = \frac{1}{d+2\gamma}\int_0^{+\infty} M_B^t(\Delta_k u)(x) t\, dt.$$

Now, using the relations (6.4), (2.4) and applying Fubini's theorem, we can see that

$$\begin{aligned}\frac{1}{d+2\gamma}\int_0^{+\infty} M_B^t(\Delta_k u)(x) t\, dt &= \frac{1}{d_k}\int_0^{+\infty} t^{1-d-2\gamma}\left(\int_{\mathbb{R}^d} \Delta_k u(y) h_k(t,x,y)\omega_k(y)dy\right) dt\\ &= \int_{\mathbb{R}^d} N_k(x,y)\Delta_k u(y)\omega_k(y)dy = N_k[\mu](x),\end{aligned}$$

where $d\mu(y) = \Delta_k u(y)\omega_k(y)dy$ is the Δ_k-Riesz measure of u (see Example 5.1).
Hence, letting $r \longrightarrow +\infty$ in the relation (2.16) with $f = u$, we deduce that

$$u(x) = h(x) - N_k[\mu](x).$$

In particular, for all $x \in \mathbb{R}^d$, $N_k[\mu](x) \leq a - u(x) < +\infty$.
Using Theorem 7.2, we deduce that u has a global Riesz decomposition on $\mathbb{R}^d$ given by $u = h - N_k[\mu]$ and the function h is D- harmonic on $\mathbb{R}^d$. Since $h \leq a$, by Liouville's theorem for bounded from above D-harmonic functions (see [8]), h is a constant. We denote again by h this constant. Furthermore, since u is D-subharmonic, we have $u(x) \leq M_S^r(u)(x) \leq h$. Then, by taking the supremum of $u(x)$ over $x \in \mathbb{R}^d$, we get $a \leq h$. Finally, we obtain $h = a$ and $u = a - N_k[\mu]$.
Let us now u be a D-subharmonic function on $\mathbb{R}^d$ and let $u_\varepsilon = u *_k \varphi_\varepsilon$ be the function defined by (4.4). We know that $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d) \cap \mathcal{SH}_k(\mathbb{R}^d)$ and its Δ_k-Riesz measure is given by $d\mu_\varepsilon(x) := \mu *_k \varphi_\varepsilon(x)\omega_k(x)dx$ (see the relation (5.4)). Moreover, as $\tau_{-x}\varphi_\varepsilon \geq 0$ and using (A.7) (recalling that $\int_{\mathbb{R}^d}\varphi_\varepsilon(y)\omega_k(y)dy = 1$), u_ε is bounded from above and we get $a_\varepsilon := \sup\ u_\varepsilon(x) \leq a := \sup u(x)$.
Now, since u is the pointwise non-decreasing limit of the sequence (u_ε) (see Theorem 4.1), the sequence of real numbers (a_ε) is also non-decreasing and $a_\varepsilon \geq a$. This proves that $a_\varepsilon = a$ for all $\varepsilon > 0$. By the first step, we conclude that

$$\forall\ x \in \mathbb{R}^d, \quad u_\varepsilon(x) = a - N_k[\mu_\varepsilon](x) \quad \text{with} \quad d\mu_\varepsilon(y) = \mu *_k \varphi_\varepsilon(y)\omega_k(y)dy.$$

Letting $\varepsilon \longrightarrow 0$ and using the relation (7.6), we deduce the desired result. □

Corollary 7.1 *1. For every $x_0 \in \mathbb{R}^d$, the zero function is the greatest D-harmonic minorant on $\mathbb{R}^d$ of the D-superharmonic function $N_k(x_0, .)$.*

2. Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ such that $N_k[\mu](x) < +\infty$ for at least one x. Then the zero function is the greatest D-harmonic minorant on $\mathbb{R}^d$ of the D-superharmonic function $N_k[\mu]$.

3. *A function u (not identically $-\infty$) defined on $\mathbb{R}^d$ is of the form $u = -N_k[\mu]+h$ where $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ and h is a D-harmonic function on $\mathbb{R}^d$ if and only if $u \in \mathcal{SH}_k(\mathbb{R}^d)$ and u has a D-harmonic majorant on $\mathbb{R}^d$. In this case, h is the least D-harmonic majorant of u on $\mathbb{R}^d$.*

Proof: By taking $\mu = \delta_{x_0}$, the statement 1) is a particular case of 2).
2) Let h be a D-harmonic function on $\mathbb{R}^d$ such that $h \leq N_k[\mu]$. Then the function $s = h - N_k[\mu]$ satisfies: i) $s \leq 0$ on $\mathbb{R}^d$, ii) s is in $\mathcal{SH}_k(\mathbb{R}^d)$ and iii) μ is the Δ_k-Riesz measure of s (by (6.15)). Therefore, by Theorem 7.3, we have

$$s = \sup_{\mathbb{R}^d} s - N_k[\mu] = h - N_k[\mu].$$

Thus, $h = \sup_{\mathbb{R}^d} s$ and by i) we must have $h \leq 0$. This proves 2).
3) Suppose that $u = -N_k[\mu] + h$. Clearly $u \in \mathcal{SH}_k(\mathbb{R}^d)$ and $u \leq h$. Now, let h_1 be a D-harmonic function on $\mathbb{R}^d$ such that $u = -N_k[\mu]+h \leq h_1$. This implies that $h-h_1 \leq N_k[\mu]$. Thus, by the statement 2), we obtain $h \leq h_1$. This proves that h is the least D-harmonic majorant of u on $\mathbb{R}^d$.
Conversely, assume that $u \in \mathcal{SH}_k(\mathbb{R}^d)$ and it has a D-harmonic majorant h_1 on $\mathbb{R}^d$. Then the function $u - h_1$ is nonpositive and D-subharmonic on $\mathbb{R}^d$. Therefore, by Theorem 7.3,

$$\forall\, x \in \mathbb{R}^d, \quad u(x) - h_1(x) = a - N_k[\mu](x)$$

for some constant $a \leq 0$. Thus, for $h = a + h_1$, $u = h - N_k[\mu]$ is the global Riesz decomposition of u and clearly we have $h \leq h_1$. □

Now, we will give a necessary and sufficient condition for $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ to be the Δ_k-Riesz measure of a bounded from above D-subharmonic function on $\mathbb{R}^d$.

Proposition 7.1 *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$. Then μ is the Δ_k-Riesz measure of a bounded from above D-subharmonic function on $\mathbb{R}^d$ if and only if there exists $x_0 \in \mathbb{R}^d$ such that*

$$\int_1^{+\infty} t^{1-d-2\gamma} n_k(t, x_0)dt < +\infty \quad \textit{with} \quad n_k(t, x_0) := \int_{\mathbb{R}^d} h_k(t, x_0, y)d\mu(y). \tag{7.7}$$

Remark 7.2 *In classical case (k=0), we have $n_0(t, x_0) = \mu[B(x_0, t)]$ and we can always assume $x_0 = 0$ by replacing the subharmonic function u of Δ-Riesz measure μ by its translate $u(x_0 + .)$ ([11], Theorem 3.20). But, if $k \neq 0$ this is not possible for at least two reasons. Firstly, the Dunkl translations act only on some functional spaces and not on sets. Secondly, they are not always positive operators. In fact, even if $u \in \mathcal{C}^\infty(\mathbb{R}^d) \cap \mathcal{SH}_k(\mathbb{R}^d)$ (i.e. $\Delta_k u \geq 0$), we don't have necessarily $\tau_x[\Delta_k u] \geq 0$ and thus $\tau_x u$ is not necessarily in $\mathcal{SH}_k(\mathbb{R}^d)$.*

Proof of Proposition 7.1: Let $u \in \mathcal{SH}_k(\mathbb{R}^d)$ bounded from above with Δ_k-Riesz measure μ. By Theorem 7.3, u is of the form $u = \sup_{\mathbb{R}^d} u - N_k[\mu]$. This proves that $-N_k[\mu] \in \mathcal{SH}_k(\mathbb{R}^d)$. Using (6.4) and Fubini's theorem, we obtain for almost every $x \in \mathbb{R}^d$

$$\textstyle\int_1^{+\infty} t^{1-d-2\gamma} n_k(t, x)dt \leq \int_0^{+\infty} t^{1-d-2\gamma} n_k(t, x)dt = d_k N_k[\mu](x) < +\infty.$$

Conversely, let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ satisfying (7.7) for some $x_0 \in \mathbb{R}^d$. We will partially follow the proof of Theorem 3.20 in [11]. Let $u(x) = -N_k[\mu](x)$. Then, by (6.15), it is enough to prove that $u \in \mathcal{SH}_k(\mathbb{R}^d)$. We can write

$$u(x) = -\int_{B^W(x_0,1)} N_k(x,y)d\mu(y) - \int_{\mathbb{R}^d\setminus B^W(x_0,1)} N_k(x,y)d\mu(y) := u_1(x) + u_2(x).$$

From Proposition 6.6, the function $u_1 \in \mathcal{SH}_k(\mathbb{R}^d)$. For $n \in \mathbb{N}$ with $n > 1$, we consider

$$v_n(x) = -\int_{B^W(x_0,n)\setminus B^W(x_0,1)} N_k(x,y)d\mu(y).$$

Again by Proposition 6.6, the function $v_n \in \mathcal{SH}_k(\mathbb{R}^d)$. Moreover, we see that u_2 is the pointwise decreasing limit of v_n on $\mathbb{R}^d$ as $n \to +\infty$. By (6.4) and Fubini's theorem, we have

$$\begin{aligned} v_n(x_0) &= -\frac{1}{d_k}\int_0^{\infty} t^{1-d-2\gamma}\int_{B^W(x_0,n)\setminus B^W(x_0,1)} h_k(t,x_0,y)d\mu(y)dt \\ &= -\frac{1}{d_k}\int_1^{\infty} t^{1-d-2\gamma}\int_{B^W(x_0,n)\setminus B^W(x_0,1)} h_k(t,x_0,y)d\mu(y)dt \\ &\geq -\frac{1}{d_k}\int_1^{\infty} t^{1-d-2\gamma} n_k(t,x_0)dt, \end{aligned}$$

where in the second equality, the integral in t variables has been decomposed on $]0,1[$ and $]1,+\infty[$ and then we have used $\forall\, t \leq 1$, supp $h_k(t,x_0,.) \subset B^W(x_0,t) \subset B^W(x_0,1)$. Letting $n \to +\infty$ and using our hypothesis (7.7), we deduce that $u_2(x_0) > -\infty$. Consequently, by Proposition 3.3, $u_2 \in \mathcal{SH}_k(\mathbb{R}^d)$. Thus, since $u = u_1 + u_2$, $u \in \mathcal{SH}_k(\mathbb{R}^d)$. $\square$

A Annex: The Dunkl transform and Dunkl's translation operators

• The Dunkl transform of a function $f \in L^1(\mathbb{R}^d, m_k)$ is defined by

$$\mathcal{F}_k(f)(\lambda) := \int_{\mathbb{R}^d} f(x)E_k(-i\lambda,x)\omega_k(x)dx, \quad \lambda \in \mathbb{R}^d, \tag{A.1}$$

where $E_k(x,y) := V_k(e^{\langle x,.\rangle})(y)$, $x, y \in \mathbb{R}^d$, is the Dunkl kernel which is analytically extendable to $\mathbb{C}^d \times \mathbb{C}^d$ and satisfies the following properties: for all $x \in \mathbb{R}^d$, $y \in \mathbb{C}^d$, all $\lambda \in \mathbb{C}$ and all multi-indices $v \in \mathbb{N}^d$ (see [4], [6], [14] and [23])

$$E_k(x,y) = E_k(y,x),\ E_k(x,\lambda y) = E_k(\lambda x,y),\ \left|\partial_y^v E_k(x,y)\right| \leq \|x\|^{|v|}\max_{g\in W} e^{Re\,\langle gx,y\rangle}. \tag{A.2}$$

It is well known (see [5] and [14]) that the Dunkl transform $\mathcal{F}_k$ is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$ (the Schwartz space) onto itself and its inverse is given by

$$\mathcal{F}_k^{-1}(f)(x) = c_k^{-2}\int_{\mathbb{R}^d} f(\lambda)E_k(ix,\lambda)\omega_k(\lambda)d\lambda, \quad x \in \mathbb{R}^d, \tag{A.3}$$

where c_k is the constant given by (1.9). Moreover, the following Plancherel theorem holds: The transformation $c_k^{-1}\mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L^2(\mathbb{R}^d, m_k)$ and we have $\|c_k^{-1}\mathcal{F}_k(f)\|_{L^2(\mathbb{R}^d,m_k)} = \|f\|_{L^2(\mathbb{R}^d,m_k)}$, $f \in L^2(\mathbb{R}^d, m_k)$.

• The Dunkl translation operators $\tau_x, x \in \mathbb{R}^d$, are defined on $\mathcal{C}^\infty(\mathbb{R}^d)$ by (see [27])

$$\forall\, y \in \mathbb{R}^d, \quad \tau_x f(y) = \int_{\mathbb{R}^d} V_k \circ T_z \circ V_k^{-1}(f)(y) d\mu_x(z), \tag{A.4}$$

where T_x is the classical translation operator given by $T_x f(y) = f(x + y)$. If $f \in \mathcal{S}(\mathbb{R}^d)$, $\tau_x f \in \mathcal{S}(\mathbb{R}^d)$ and using the Dunkl transform for all $y \in \mathbb{R}^d$ we have (see [27]):

$$\tau_x f(y) = \mathcal{F}_k^{-1}[E_k(ix, .)\mathcal{F}_k(f)](y) = c_k^{-2} \int_{\mathbb{R}^d} \mathcal{F}_k(f)(\lambda)E_k(ix, \lambda)E_k(iy, \lambda)\omega_k(\lambda)d\lambda.$$

The operators $\tau_x, x \in \mathbb{R}^d$, satisfy the following properties:

1) For all $x \in \mathbb{R}^d$, the operator τ_x is continuous from $\mathcal{C}^\infty(\mathbb{R}^d)$ into itself.

2) For all $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and all $x, y \in \mathbb{R}^d$, we have

$$\tau_x f(0) = f(x), \quad \tau_x f(y) = \tau_y f(x). \tag{A.5}$$

3) The Dunkl-Laplace operator Δ_k commutes with the Dunkl translations i.e

$$\tau_x(\Delta_k f) = \Delta_k(\tau_x f), \quad x \in \mathbb{R}^d, \quad f \in \mathcal{C}^\infty(\mathbb{R}^d). \tag{A.6}$$

4) For any $f \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \tau_x f(y)\omega_k(y)dy = \int_{\mathbb{R}^d} f(y)\omega_k(y)dy \tag{A.7}$$

5) Let $f \in \mathcal{S}(\mathbb{R}^d)$ be radial. Then we have (see [8], Lemme 3.1)

$$\tau_{-x} f(y) = \tau_{-y} f(x) \tag{A.8}$$

6) Let $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $g \in \mathcal{D}(\mathbb{R}^d)$. Then, we have (see [8], Proposition 2.1):

$$\forall\, x \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} \tau_x f(y)g(y)\omega_k(y)dy = \int_{\mathbb{R}^d} f(y)\tau_{-x} g(y)\omega_k(y)dy. \tag{A.9}$$

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